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EVALUATING INVESTMENT OPPORTUNITIES UNDER DIFFERENT MODEL DYNAMICS: SOME MANAGERIAL INSIGHTS

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Abstract

The Net Present Value is the most well known measure of project valuation for managers. However it requires an investment decision made in the moment as well as the cash outflow of the investment. However, a manager has different levels of flexibility in the exercise of his functions that the classic Net Present Value valuation does not take in account. An investment can be delayed to a pre-committed date, can have the decision delayed by a certain or an endlessly period of time, and can be reverted. Despite not applicable to all parameters, the numerical analysis made in this thesis has a pretty straight-forward conclusion, the higher the flexibility a manager can dispose, the value of the project for the manager rises.

A project value is not only affected by its parameters and by the flexibility disposed to the manager. The model dynamic in which a project is calculated is also a very important tool for managers to consider. The most used model dynamic to value real options is the Geometric Brownian Motion, which assumes constant volatility. Constant volatility is not a legit assumption to take, since a wide range of assets and markets do not have constant volatility. To overcome this flaw, the Constant Elasticity of Variance diffusion model is considered in this thesis. Numerical analysis made in this thesis proves that a manager is exposed to real options valuation errors by assuming constant volatility.

Key-words: Real Options, Flexibility, Geometric Brownian Motion, Constant Elasticity of Variance diffusion

JEL Classification System: G13; G31; D81; D92; C61

Resumo

O Valor Atualizado Líquido é a mais conhecida medida de avaliação de projetos para gestores. No entanto, requer uma decisão de investimento imediatamente assim como o cash outflow do investimento. Contudo, um gestor tem diferentes níveis de flexibilidade no exercício das suas funções, flexibilidade essa que a avaliação com o clássico Valor Atualizado Líquido não tem em conta. Um investimento pode ser adiado para uma data pré-acordada, pode ser adiada a decisão até um certo ou um indefinido período de tempo, e pode ser revertido. Apesar de não ser aplicável para todos os parâmetros, a análise numérica feita nesta tese tem uma conclusão clara, quanto maior a flexibilidade que um gestor dispõe, maior o valor do projeto para o gestor.

O valor de um projeto não é só afetado pelos seus parâmetros e pela flexibilidade à disposição do gestor. A dinâmica do modelo no qual o projeto é calculado é também um fator muito importante para o gestor ter em conta. O modelo mais usado para avaliar opções reais é o movimento Browniano geométrico, que assume uma volatilidade constante. Volatilidade constante não é uma assunção legítima de fazer, visto que um largo espectro de ativos e mercados não têm volatilidade constante. Para superar esta falha, o modelo de difusão da Constante Elasticidade da Variância é considerado nesta tese. A análise numérica feita nesta tese prova que um gestor na avaliação de opções está exposto a erros por assumir a volatilidade constante.

Palavras-chave: Opções Reais, Flexibilidade, Movimento Browniano Geométrico, Difusão da Constante Elasticidade da Variância

Sistema de Classificação JEL: G13; G31; D81; D92; C61

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List of Abbreviations

NPV - Net Present Value

GBM – Geometric Brownian Motion

CEV – Constant Elasticity of Variance

1. Introduction

In a daily basis, managers face investment decisions. In order to decide whether the manager invests or not in a project, it is universally common to use the Net Present Value, the most basic and traditional rule taught in every business school. The process of decision based on the NPV is pretty simple, if the present value of the future cash-flows exceeds the investment (and other cash outflows, then the project should be taken, if not, the project should be automatically dropped. The numbers of all the parameters are known in advance, being the NPV a deterministic model with only one possible result. This rule assumes that the project can't be reverted to an idle state in the future, remaining the manager passive to any possibility of market circumstances change. In reality managers have more options.

A manager will not continue a project if the market conditions turn so bad to the point that running a project generates losses to a firm. In the same way, if market conditions turn out unexpectedly good, a manager surely can augment the size of the gains. These real possibilities are not included in the classic NPV valuation of projects, and it is legit that the valuation of a project includes this possibilities.

Keswani and Shackleton (2006) show in their paper how the real options approach augments the NPV of a project. They consider various hypotheses a manager can have, such as delaying the project to a pre-committed date, delaying the decision to invest in the project during certain time as well as possibility to revert the investment decision. They make most of their calculus based on the pricing model of Black and Scholes (1973). The deterministic model of NPV is no longer the more appropriated to value project value. The stochastic processes considered by Black and Scholes (1973) take in consideration the random factor that the evolution of the market circumstances is.

Keswani and Shackleton (2006) were not the only ones with work done in the real options pricing field. Dias and Nunes (2011) have a different approach. Keswani, Shackleton and many others price real option based on the Geometric Brownian Motion, a method that assumes constant volatility. Dias and Nunes (2011) developed pricing formulas under the constant elasticity of variance diffusion process by Cox (1975). Dias and Nunes show the errors that managers can be exposed in investment and disinvestment decisions by ignoring the market evidence of non constant volatility for a wide range of markets and assets.

Various authors call the attention that some assets or markets can evidence a mean-reverting process, this is, an asset can have a tendency to revert to a long run value, and should be calculated according with that phenomenon. Tsekrekos (2010) developed pricing formulas based on Sarkar (2003) work on price reverting processes. Tsekrekos shows the errors a manager can be exposed by the more easy to compute Geometric Brownian Motion in the presence of an asset with a mean-reverting tendency.

In this thesis, a more in-depth look at the different investment opportunities under different model dynamics is developed.

The thesis is organized by chapters. Chapter 2 shows theory about valuation under deterministic processes, Chapter 3 shows theory behind the stochastic process, the Geometric Brownian Motions as well as valuation formulas for different kind of real options under stochastic process, Chapter 4 shows the theory about pricing real options under the Constant Elasticity of Variance diffusion, Chapter 5 gives a better comprehension of mean-reverting processes, Chapter 6 provides numerical analysis of manager flexibility under the Geometric Brownian Motion process, Chapter 7 provides numerical analysis of the perpetual American-style option under the Constant Elasticity of Variance diffusion process. Chapter 8 concludes.

2. Deterministic processes

2.1. Classic NPV

In order to evaluate an investment project, the Net Present Value (hereafter, NPV) is without a doubt the most well known method to do it. The mechanics behind the method are pretty simple. All the future cash inflows (and out follows) are discounted to the present at an appropriate discount rate (typically found in the financial markets). The present value of these future cash-flows are added and then the initial cash outflow (commonly known as investment) is subtracted, that is

$$\mathrm{NPV} = \frac{\sum_{i=0}^{n} \mathrm{CFi}}{(1+\mu)^{i}} - \bar{X} \qquad (1)$$

where CF_i, is the cash-flow at time i, μ is the risk-adjusted discount rate of the project, i the time to discount to the present and $\bar{X}$ the investment value of the project today.

In the case of a perpetuity, the NPV will be (one must require that r > g for convergence reasons):

$$\mathrm{NPV} = \frac{v_0}{\mu - g} - \bar{X} \qquad (2)$$

where v_0 is the cash "inflow" that the project has in time 0, and g is the annual continuously growth rate of v_0.

With the NPV method, the decision to invest occurs when NPV > 0. The investment threshold $\bar{X}$ is equal to all the future cash flows discounted to the present, that is

$$\bar{X} = \frac{\sum_{i=0}^{n} \mathrm{CFi}}{(1+\mu)^{i}} \quad \text{or} \quad \bar{X} = \frac{v_0}{\mu - g} \qquad (3)\ (4)$$

where $\bar{X}$ is the threshold where investors should start to invest.

However, with the NPV method, we are only analyzing the value of a project whose start date is immediate and there is no possibility to postpone the project. Also, NPV assumes that after a decision to commit to a project is made, the manager can't do anything even if market circumstances change. In real life cases, managers can possibly postpone the decision to invest and wait for a change of the economic scenario, and can also divest after the investment has been made. This kind of flexibility naturally gives more value to a project than the classic NPV approach. Keswani and Shackleton (2006) suggest different project value methods for different situations (or in practice, different types of real options).

2.2. Forward Start NPV

With the classic NPV method, to obtain that metric value, the investor needs to start the project today. But if the investor has the possibility to invest in the project at a pre-committed forward start time? This delay of the project would be beneficial to the project.

The delay of the project will turn the present value of the costs and the revenues lower, so the delay of the project could have benefits, if the rate at which the costs fall is higher than the revenue fall.

Keswani and Shackleton (2006) suggest the following equation to determine the NPV of a forward start project:

$$NPV = V_0 e^{-qT} - \overline{X} e^{-rT} \tag{5}$$

where V_0 is the risk discounted sum of cash flows, q is the dividend yield (cash-flow yield on a project), r is the risk-free rate, and T is the number of years that an investor decides to delay the project.

The authors argue this equation makes sense, since the present value of the costs (investment) are known, so they fall in proportion to the risk-free rate, and the present value of the cash-flows fall in proportion to the dividend yield (the yield of cash the investor is not receiving by delaying the project).

3. Stochastic Processes

The previous cases of determining the value of a project (Classic NPV and Forward Start NPV) are deterministic processes, this is, there is no random factor on this methods and the output will be always the same once initial conditions are known. In a stochastic model, at least part of a variable evolution has a random factor. From this point of the thesis to beyond stochastic processes will be considered.

3.1. Geometric Brownian Motion

The Geometric Brownian Motion is a continuous-time stochastic process commonly used in the field of mathematical finance, especially for valuing financial options.

To better understand this concept, it is necessary to describe first what is a Brownian Motion. Dixit and Pindyck (1994) refer Brownian Motion as a continuous-time stochastic process that must satisfy three conditions:

- It satisfies the Markov property, i.e. that all the forecasts of the process depend only on the current value, and not by what happened before;
- It has independent increments, which means that the probably distribution in one time interval in the process is not related with the probably distribution of another time interval in the process;
- The changes in the process are normally distributed, with the variance increasing linearly with the finite time interval

In a Brownian Motion, the variance of the change grows linearly with the time horizon. The increment of a Brownian Motion (dz) is represented in continuous time by:

$$dz = \epsilon_t \sqrt{dt} \qquad (6)$$

where ϵ_t is a variable normally distributed with mean 0 and unit standard deviation, and dt is the change in time.

However, it is impossible that a stock price falls below 0, so it is unreasonable to model prices based in a normal distribution process. We can assume that stock prices are lognormally distributed, so that stock prices never fall below 0. The Brownian motion serves as a base to more realistic and commonly used models as the Geometric

Brownian Motion. As a special case of a Brownian Motion, the Geometric Brownian Motion is represented by the following equation:

$$dx = \alpha\, x\, dt + \sigma x\, dz \tag{7}$$

where α and σ are constants (α is the drift or growth parameter and σ is the variance parameter) and dx represents a continuous-time stochastic process $x(t)$. Absolute changes in x are lognormally distributed. Computing paths of values (that can be stock prices as well as many other economic variables) with determined drift (growth rate) and variance can be executed with GBM with the following equation (equation that gives the values for a specific time):

$$x_t = (\alpha + 1)x_{t-1} + \ \sigma x_{t-1}\epsilon_t \tag{8}$$

In this equation, $(\alpha + 1)x_{t-1}$ is the known part of the process while $\sigma x_{t-1}\epsilon_t$ is the random part.

We can forecast a value with the GBM model, although, we can only use to forecast the known part of the process. To any value in certain time T, the forecast of x can be represented by the following equation:

$$x_{t+T} = (\alpha + 1)^T x_t \tag{9}$$

3.2. Transition of financial options to real options

In the financial markets, financial options are a product available for trading. This product gives the buyer an option (and the seller an obligation) to buy or sell a financial product such as shares (could be also futures, indexes, currencies, etc.), until the expiration date of the contract.

Real Options work in the same way, but instead of having financial products we have real assets or projects.

Investment Opportunity	Variable	Call Option
PV of a project's expected cash flows	V_t	Stock price
Investment cost	X	Strike price
Length of time the decision can be deferred	τ	Time to expiration
Project value uncertainty	σ	Stock's volatility
Riskless interest rate	r	Riskless interest rate
Rate of return shortfall	q	Dividend yield

Figure 1 Components of an investment project stated as components of financial options valuation (Source: *"Real Options Valuation"* by Dias(2011))

Comparing financial with real options, in the case of financial options we work with the current value of the financial product which represents in real options the present value of the expected cash-flows, the exercise price corresponds to the investments costs since in both occasions that is the price to "pay" in order to pursue the investment, the expiration date corresponds to the time until a project opportunity disappears, and the uncertainty of the financial product corresponds to the uncertainty of a given project. The value of the risk free rate and dividend yield correspond the same in the two occasions.

3.3. European Option NPV

Suppose an investor has a project in which he can delay the decision to invest, to a given future time. When that time approach, investor can see if it is a good idea to advance with the project, or if it is better to dump the project. In financial options, that is equivalent to a European call option. If at the time of the decision the present value of costs is higher than the present value of revenues, the investor should dump the project (out-of-the-money option), and if is lower it should accept the project (in-the-money option).

In the formula created by Black, Scholes and Merton (1973), the investment would be the strike price (known at the beginning), and the present value of the revenues the spot price (which can vary). Thus,

$$NPV = V_0 e^{-qT} N(d_1) - \bar{X} e^{-rT} N(d_2) \tag{9}$$

$$d_1 = \frac{\ln\left(\frac{V_0}{\bar{X}}\right) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{10}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{11}$$

The NPV of a European call option is naturally higher than the classic NPV, since in the European Call Option, the investor has the flexibility of not accepting the project if market conditions deteriorate.

3.4. Finite American Option NPV

A finite American option is appropriate when a manager can invest in a project at any time inside of a pre-defined maturity. In order to decide when it is the better time to invest and what is the NPV of its investment, the manager needs to compare at a given time for all the possible times if it is better to get the difference between the present value of the revenue of the project and the investment of it or if it is better to keep the option. As a proxy for the finite American option NPV, it will be considered 250 time steps a year organized in a binomial tree in order to be possible to calculate the NPV of a finite American Option.

3.5. Perpetual American Option NPV

The NPV of a perpetual American option is appropriate for projects that can be delayed endlessly in time. The investor has all the time and possibility to advance or not with a given project. This type of NPV can be applied to monopoly companies.

An investor should take the project if the value of the project hits the threshold $\overline{V}$.The value of $\overline{V}$ is time independent in this case because there is not a limit of time to take the option. A higher value of $\overline{V}$ eventually raises the payoff exercising the option. The following equation shows the optimal value of V that maximizes the perpetual value of the option, dealing with presence of this value/time tradeoff:

$$\overline{V} = \frac{a}{a-1}\,\overline{X} \tag{12}$$

The a parameter is an elasticity constant that we can obtain from the following fundamental quadratic (b parameter which is the "negative" part is also calculated from this quadratic and will be useful in the next chapters of the thesis):

$$a, b = \frac{1}{2} - \frac{(r-q)}{\sigma^2} \pm \sqrt{\left(\frac{(r-q)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{13}$$

So, using the optimal value of the threshold $\overline{V}$, the optimal NPV is calculated by the following formula:

$$NPV = V_0\left(\frac{V_0}{\overline{V}}\right)^{a-1} - \overline{X}\left(\frac{V_0}{\overline{V}}\right)^{a} \tag{14}$$

The value of the American Perpetual Option will be naturally higher than the options analyzed before, since in this case exists more flexibility.

3.6. Reversible NPV (Perpetual Case)

A perpetual costly reversible NPV it is an NPV of a situation when a manager can invest in a project endlessly in time, and then can divest from the project endlessly in time at a given cost (or income). A perpetual costless reversible it is the same as the costly reversible NPV with only the difference of recovering all the initial investment of the project. As we have a divestment opportunity, we need to have a divest revenue (or cost if $\underline{X}$ is negative), which is represented by $\underline{X}$, and a divest threshold $\underline{V}$. As we did for $\overline{V}$ in the perpetual American option chapter, we can calculate $\underline{V}$ as the optimal one time divest value of an active project with the following equation:

$$\underline{V} = \frac{b}{b-1}\,\underline{X} \qquad (15)$$

However, for calculating the V thresholds of a perpetual reversible option, we need to compute these thresholds simultaneously. The investment and disinvestment values are the key factors that determine the threshold values.

Considering the α parameter (α = $\overline{X}$ / $\underline{X}$), we can have the following cases of reversibility:

- α = 0, the investment is irreversible (the investor cannot undo the investment decision)
- 0 < α < 1, the investment is partially reversible, that is, costly reversible
- α = 1, the investment is completely reversible, that is, costless reversible

As reversibility is always available for the manager, in the idle state the manager will always compare the value of the option to open plus the investment value with the value of the project in the operating state plus the option to close. When this values match we have an optimal $\overline{V}$. In the active state, the manager will compare the value of the project open plus the option to close, with the value of disinvestment plus the value of the option to open. When this values match we have an optimal $\underline{V}$. So, with these two conditions, we can express two value matching equations (respectively the following first and third equation). To get the optimal values of the thresholds, we will also need two associated smooth pasting conditions (respectively the following second and fourth equation), which are derivatives from the value matching equations, that is,

$$-A_1\overline{V}^a + B_2\overline{V}^b + \overline{V} = \overline{X} \quad (16)$$

$$-aA_1\overline{V}^{a-1} + bB_2\overline{V}^{b-1} + 1 = 0 \quad (17)$$

$$-A_1\underline{V}^a + B_2\underline{V}^b + \underline{V} = \alpha\overline{X} \quad (18)$$

$$-aA_1\underline{V}^{a-1} + bB_2\underline{V}^{b-1} + 1 = 0 \quad (19)$$

Where A_1 and B_2 are constants necessary to compute in order to get the optimal values of the investment and disinvestment threshold.

Once we have the value of the thresholds and the constants, the NPV in the reversible case follows the following formula:

$$NPV = A_1 * {V_0}^a \quad (20)$$

4. Constant Elasticity of Variation Diffusion

Since the Geometric Brownian Motion possess a setup to with the undesirable feature of assuming a constant volatility, Dias and Nunes (2011) created a formula to price real options under the Constant Elasticity of Variation Diffusion process of Cox (1975).

4.1. Setup of the model

Dias and Nunes model real asset prices based on the following one dimensional diffusion process:

$$\frac{dV_t}{V_t} = (r - q)dt + \sigma(t, V)dW_t^{\mathbb{Q}} \tag{21}$$

where r represents the instantaneous riskless rate, q represents the dividend yield , $\sigma(t, V)$ is the instantaneous volatility per unit of time of asset returns, and $W_t^{\mathbb{Q}}$ is a standard Brownian motion

The CEV (constant elasticity of variance) process by Cox(1975) is expressed in the following stochastic differential equation:

$$dV_t = (r - q)V_t dt + \delta V_t^{\frac{\beta}{2}} dW_t^{\mathbb{Q}} \tag{22}$$

The equation has the following volatility function:

$$\sigma(t, V) = \delta V_t^{\frac{\beta}{2}-1} \tag{23}$$

β is the key parameter of this setup. If $\beta = 2$, the model becomes a Geometric Brownian Motion. If $\beta > 2$, the volatility and the asset value are positively related. When $\beta < 2$, the volatility and the asset value are inversely related. The models of Cox and Ross (1976) can also be specified by Equation 22, where the absolute diffusion is represented by $\beta = 0$, and the square-root diffusion is represented by $\beta = 1$, however this cases will not be treated in this thesis.

The model parameter δ is a positive constant representing the scale parameter that fixes the initial time t instantaneous volatility to be equal across CEV models with different β parameter.

4.2. Option Pricing

Dias and Nunes (2011) offer closed-form solutions in order to compute the value and threshold of perpetual American-style Options for the case where the risk-free rate is equal to dividend yield as well as when r and q are different. By maximizing the limits of the equations of both cases, it is obtained the entry or exit threshold of the option depending if it is a call or a put.

4.2.1 Perpetual American-style Option when r ≠ q

$$\lim_{T\uparrow\infty} \Theta_t(V,X,T;\phi) = \phi(X - V_\infty)\left(\frac{V_t}{V_\infty}\right)^{\eta(\phi)} exp\{\eta(\phi)[x(V_t) - x(V_\infty)]\} * \frac{M_{\phi(\beta-2)}\left[\eta(\phi) + (-1)^{\eta(\phi)}\alpha, \frac{\beta-1-2\eta(\phi)}{\beta-2}, (-1)^{\eta(\phi)}x(V_t)\right]}{M_{\phi(\beta-2)}\left[\eta(\phi) + (-1)^{\eta(\phi)}\alpha, \frac{\beta-1-2\eta(\phi)}{\beta-2}, (-1)^{\eta(\phi)}x(V_\infty)\right]} \quad (24)$$

with φ = -1 for an American Call and φ = 1 for an American Put φ

$$V_\infty = \begin{cases} \overline{V} \Leftarrow \phi = -1 \\ \underline{V} \Leftarrow \phi = 1 \end{cases} \quad (25)$$

$$\eta(\phi) := \begin{cases} 1_{\{r>q\cap\beta<2\}} & \Leftarrow \quad \phi = +1 \\ 1 - 1_{\{r>q\cap\beta>2\}} & \Leftarrow \quad \phi = -1 \end{cases} \quad (26)$$

$$\alpha := \frac{r}{(\beta-2)(r-q)} \quad (27)$$

$$x(V) = \frac{2(r-q)}{\delta^2(\beta-2)} V^{2-\beta} \quad (28)$$

and

$$M_\lambda(a,b,z) := \begin{cases} M(a,b,z) & \Leftarrow \quad \lambda > 0 \\ U(a,b,z) & \Leftarrow \quad \lambda < 0 \end{cases} \quad (29)$$

Where $M(a,b,z)$ and $U(a,b,z)$ are the confluente hypergeometric functions defined in Abramowitz and Stegun (1972, eq. 13.1.2 and 13.1.3).

4.2.2 Perpetual American-style Option when r = q

$$\lim_{T\uparrow\infty}\Theta_{\mathrm{t}}(V,X,T;\phi)=\phi(X-V_{\infty})\sqrt{\frac{V_t}{V_\infty}}\frac{I_{\frac{1}{|\beta-2|};\phi(\beta-2)}\left[\varepsilon(V_t)\sqrt{2r}\right]}{I_{\frac{1}{|\beta-2|};\phi(\beta-2)}\left[\varepsilon(V_\infty)\sqrt{2r}\right]} \tag{30}$$

where

$$\varepsilon(V):=\frac{2V^{1-\beta/2}}{\delta|\beta-2|} \tag{31}$$

and

$$I_{v;\lambda}(z):=\begin{cases}I_v(z) & \Leftarrow \quad \lambda>0\\ K_v(z) & \Leftarrow \quad \lambda<0\end{cases} \tag{32}$$

Where $I_v(z)$ and $K_v(z)$ are the modified Bessel functions defined in Abramowitz and Stegun (1972, page 375).

5. Mean-Reverting process

In financial historical data, many financial and economic assets and variables have a tendency to revert to a value in the long term.

From the point of view of a manager the success of many firms can directly derive from a price of a commodity whose price evolution is uncertain. Logically, entry and exit decisions need to be done by the manager with the price evolution in mind, as well as the entry or exit costs of a given project. As Tsekrekos (2010) argues, if the price of a good in some determined market rises, more companies will enter that market and existing ones will reinforce the position in the market until the market is crowded enough to no longer take advantage of a risen price. Then, as the price of a good can fall, the supply is too high for the demand in terms of value and existing companies will tend to exit the market, since the market is not enough for all the companies in operation, until the price rise again. As the price rise and fall in this situation, it is intuitive that the price tend to revert to a value between these extremes instead of following a Geometric Brownian motion. So for cases like this, a manager could incur in big errors of determining the value, entry and exit threshold of a project if uses the Geometric Brownian Motion instead of a mean-reverting process.

Sarkar (2003) proposed the following model for a mean-reverting process:

$$dP = \kappa(\theta - P)dt + \sigma P dz \qquad (33)$$

where P is the output price, κ is the speed of the reversion to the long run mean price, and θ is the long run mean price value.

If $\theta = 0$ the process of Sarkar (2003) turns into a geometric Brownian motion, and for $\kappa = 0$, the equation turns into a geometric Brownian motion without drift.

In his paper, Tsekrekos (2010) used the process of Sarkar (2003) in order to compare the effect of a mean reverting process to a geometric Brownian Motion, in the case where the investor can invest and revert the investment endlessly in time. He concludes that by following a mean reverting process, *“the composition of an industry experiences less frequent changes under mean reversion”*. In a practical way that means that the values of entry threshold are higher and the values of exit threshold are lower in a mean reversion process in relation to the geometric Brownian motion. This conclusion makes economical sense, since if a company wants to enter in a market, is because that price of that market is high (higher than the long run price level), and although with the GBM the manager enter the market in a certain entry threshold, for the same value, a mean-reverting process consider that the price will revert to a lower value, requiring a higher threshold for the manager optimally enter the market. The same conclusion can be made regarding a price lower than the long run price level and the exit threshold.

6. Case Study

6.1. Manager Value of Flexibility

In this chapter we take a closer look on how flexibility can affect the manager's decisions regarding projects/investments, following the calculation based on the Geometric Brownian Motion.

For the development of this case study, we will use, as the base case scenario, the values of the next table for the different parameters:

Table 1 Parameters and variables of the case

Variable	Symbol	Value
Horizon Time	T	10
Project Investment Cost	$\overline{X}$	2200
Project cash flow at time 0	v_0	150
Project value at time 0	V_0	2500
Project disinvestment value	$\underline{X}$	1900
Risk-free rate	r	5%
Required rate of return	μ	10%
Dividend yield	q	4%
Project capital gain	g	4%
Uncertainty of the project	σ	20%

Note: $V_0 = \frac{v_0}{\mu - g}$

Starting from the basics, I first look to the classic NPV. Using the formula, I get the following value:

$$NPV_1 = \frac{v_0}{\mu - g} - \overline{X}$$

$$NPV_1 = \frac{150}{10\% - 4\%} - 2200$$

$$NPV_1 = 300$$

This result (NPV=300) represents what a manager/investor value can get without any flexibility. To get this NPV, the investor has to make a decision on the investment now, since there is no possibility of delaying the decision or to reverse the investment. The threshold of investment ($\overline{V}$) will be naturally equal to $\overline{X}$, the project is viable once V_0 achieves 2200, the necessary value to cover the investment.

Let us assume now that the investor can delay a project to a pre-committed date. However the decision must be made today as well. In this case we have a case of a forward start NPV. Making the calculations for this case, and assuming a 10 year pre-committed date, we get the following result:

$$NPV_2 = V_0 e^{-qT} - \overline{X} e^{-rT}$$

$$NPV_2 = 2500e^{-0,04*10} - 2200e^{-0,05*10}$$

$$NPV_2 = 341,43$$

As we can see the NPV resulting from pre-commit to invest in 10 years is a better option than investing now as the investor/manager can get more 41,43 units of value. The manager can explore this because the present value of the costs of the project falls faster than the present value of the revenues. The contrary case could happen, but in that case the manager would prefer to invest in the project now and getting the NPV of 300.

Now I will assume a case where a manager can delay the project to a pre-committed date, and in that date the manager has the option to invest or not in the project according to the economic situation on that pre-committed time. This type of real option follows the logic of the European-style Option and the project must be valued like that. Making the calculations for the case we got the following NPV_3:

$$NPV_3 = V_0 e^{-qT} N(d_1) - \bar{X} e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{V_0}{\overline{X}}\right) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$d_2 = d_1$- $\sigma\sqrt{T}$

$$NPV_3 = 2500e^{-0,04*10} N(d_1) - \bar{X} e^{-0,05*10} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{2500}{2200}\right) + \left(0,05 - 0,04 + \frac{0,2^2}{2}\right) * 10}{0,2\sqrt{10}} = 0,67646$$

$d_2 = 0,67646$ - $0,2\sqrt{10} = 0,044$

$$N(d1) = N(0,67646) = 0,750627$$

$$N(d2) = N(0,044) = 0,517551$$

$$NPV_3 = 2500e^{-0,04*10} * 0,750627 - \bar{X}e^{-0,05*10} * 0,517551$$

$$NPV_3 = 567,3$$

The possibility of delaying the decision to a pre-committed date gives the manager an option worthing 267,3 units of value (NPV_3 – NPV_1).

Assuming the case where the manager can delay the decision, and invest (or not) at any point in the time until a known expiration date, such real option is equivalent to a finite American call option. Being the Finite American call option a model with complex calculations, the NPV of this case was computed on a *Matlab* file (using the binomial method, and assuming 250 steps per year, which means 2500 steps in total). Its value its equal to:

$$NPV_4 = 647,44$$

The possibility to delay to make or not the project, at any time till the next 10 years expiration date, gives the manager an option worthing 347,44 units of value (NPV_4 – NPV_1).

For the following types of options it is necessary to compute the a and b parameters of the fundamental quadratic:

$$a, b = \frac{1}{2} - \frac{(r - q)}{\sigma^2} \pm \sqrt{\left(\frac{(r - q)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$a = \frac{1}{2} - \frac{(5\% - 4\%)}{20\%^2} + \sqrt{\left(\frac{(5\% - 4\%)}{20\%^2} - \frac{1}{2}\right)^2 + \frac{2 * 5\%}{20\%^2}} = 1,85078$$

$$b = \frac{1}{2} - \frac{(5\% - 4\%)}{20\%^2} - \sqrt{\left(\frac{(5\% - 4\%)}{20\%^2} - \frac{1}{2}\right)^2 + \frac{2 * 5\%}{20\%^2}} = -1,35078$$

If the manager has the flexibility to postpone the decision to make the project endlessly in the future, that is the equivalent to a Perpetual American Option. Making the calculation for this case we have:

$$\overline{V} = \frac{a}{a-1}\,\overline{X}$$

$$\overline{V} = \frac{1{,}85078}{1{,}85078 - 1} * 2200$$

$$\overline{V} = 4785{,}86$$

$$NPV_5 = V_0 \left(\frac{V_0}{\overline{\overline{V}}}\right)^{a-1} - \bar{X}\left(\frac{V_0}{\overline{\overline{V}}}\right)^{a}$$

$$NPV_5 = 2500\left(\frac{2500}{4785{,}86}\right)^{1{,}85078-1} - 2200\left(\frac{2500}{4785{,}86}\right)^{1{,}85078}$$

$$NPV_5 = 777{,}41$$

The rise of flexibility in this case gives to the investor an option that worths 477,41 in relation to decide and invest in a project now. In relation to the European-style option case, the Perpetual American case add 210,11(NPV_5 – NPV_3) units of value.

In the previous cases, the manager only could decide if he should invest in the project or not, and when. Now, considering that after investing in a project, the manager can divest the project if is not worth it to maintain the project active under the economic circumstances. The reversible NPV can be applied to this case, and I will consider two scenarios:

- There is a costly reversible NPV, only recovering 1900 money units of the 2200 invested (alpha = 86,36%)
- There is a costless reversible NPV, the manager recovers all the money invested (alpha = 100%)

The calculation of the reversible NPV involves a very complex equation system already described in chapter 3, making these calculations on the software *Mathematica 7.0*. This system gives us not only the NPV as the investment and disinvestment thresholds.

For the costly reversible case ($0 < \alpha < 1$)we get:

$$\overline{V} = 4048$$

$$\underline{V} = 1576$$

$$NPV_6 = 835{,}34$$

For the costless reversible case (α = 1) we get:

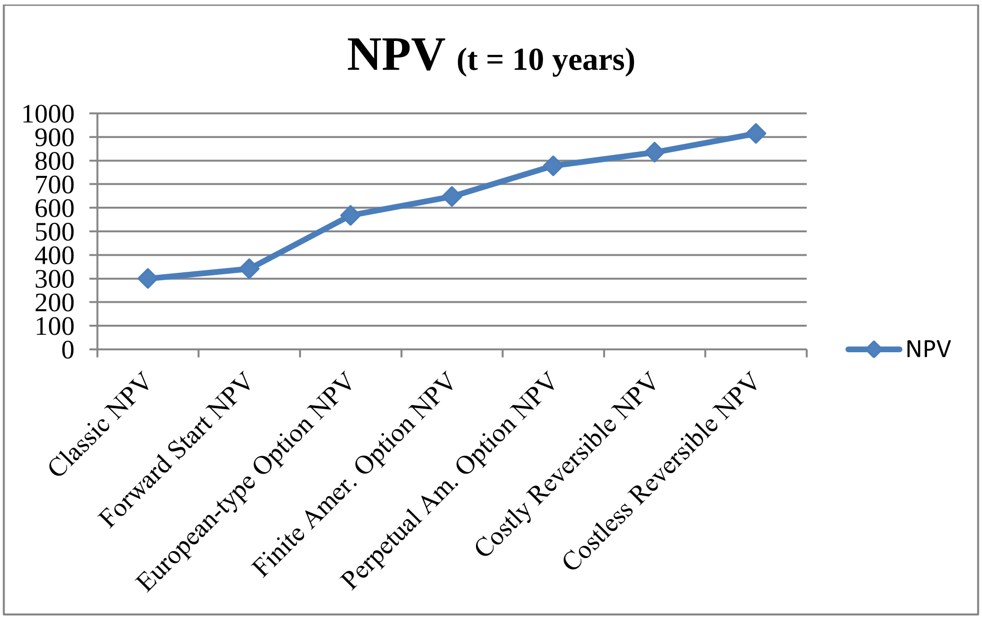
$$\overline{V} = 2750$$

$$\underline{V} = 2750$$

$$NPV_7 = 914{,}58$$

As we can see by the NPV's, another augment of flexibility will turn in an augment of value for the investor. In relation to the classic NPV, the costly reversible NPV adds 535,34 units of value, being its NPV almost the triple of the classic NPV , and the costless reversible NPV adds 614,58 units of value, being its NPV more than the triple of the classic NPV. As we can see, the entry and exits thresholds for the costless reversible are the same. This makes sense since the investment cost and the disinvestment benefit have the same value.

Graph 1 NPV in different levels of flexibility



As we can see in graph 1, the highers the degree of flexibility of the option, the higher the option value.

Table 2 Relative comparison between different levels of flexibility

	NPV	% difference in relation with Classic NPV	% difference in relation with previous NPV
Classic NPV	300,00	-	-
Forward Start NPV	341,43	13,81%	13,81%
European-style Option NPV	567,30	89,10%	66,15%
Finite American-style Option NPV	647,44	115,81%	14,13%
Perpetual American-style Option NPV	777,41	159,14%	20,07%
Costly Reversible NPV (Perpetual)	835,34	178,45%	7,45%
Costless Reversible NPV (Perpetual)	914,58	204,86%	9,49%

In table 2, we can make some relative comparisons between the different levels of flexibility. As already stated, the most flexible kind of option in this case gives to the manager more than the triple of the Classic NPV. This is the costless reversible NPV in the case gives the manager 204,86% more value than the classic NPV. Until 100% of gain value from the Classic NPV there is the Forward Start NPV and the European-style Option. Between the double and triple of the value of Classic NPV (between plus 100% and 200%) there is the Finite and Perpetual versions of American-style Option NPV, and the costly reversible NPV.

Moreover, the higher is the degree of flexibility in the option, the higher is the option value. However with the data of graph 1 and table 2, we can conclude that the NPV evolution through the different levels of flexibility is not constant. As stated in table 2, the difference in value between the forward start NPV and the European-style Option NPV is 66,15% and the difference in value between the finite and the perpetual American-style options is 20,07%. The relative difference between other degrees of flexibility do not get higher than 14,13%.

These big differences between degrees of flexibility can be explained by the different way the net present value of that options are calculated:

- The Forward-start NPV method of calculation is a deterministic process unlike the European-style Option NPV which is a stochastic process based on the Geometric Brownian Motion.
- The finite American-style option NPV is based on the Partial differential equation, unlike the perpetual American-style option NPV which is based on the Ordinary differential equation where the result is not dependent of time.

6.2. Alternative Scenarios

In order to better understand the benefits of real options for managers, it is important to analyze different scenarios assuming different parameters. For this purpose it will be considered scenarios with time until 50 years to see the evolution of different real options NPV through the years, different investment costs (1900, 2500, 2800, 3100) to see how the value of the real options react to different situations of the Classic NPV (negative drift, zero drift and positive drift), the dividend yield will be tested at 5% and 6% with the goal of having a situation where $q=r$ and other where $q<r$. A scenario with more uncertainty (standard deviation = 40%) will also be considered. This alternative scenario will also be analyzed in different models of real option later such as in the Constant Elasticity of Variance (CEV) diffusion model. In order to better understand the costless NPV, various scenarios of the alpha parameter will be analyzed.

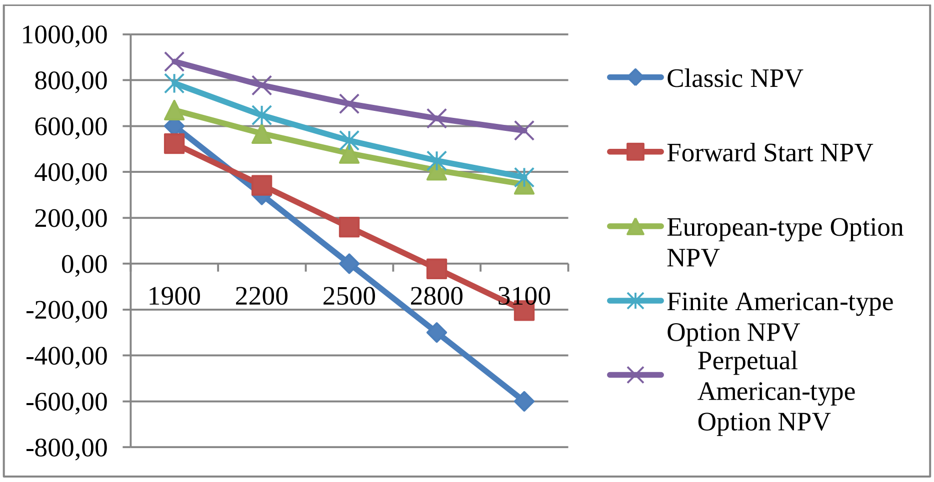
6.2.1 Investment costs

Testing different investment costs for the different kind of options, it is possible to argue that based on Graph 2 that a higher investment cost (*ceteris paribus*) will always translate to a smaller NPV in every kind of flexibility. However we can see that for an investment cost of 1900 the Classic NPV is higher than the Forward Start NPV, this means that for this investment cost, the benefits of delaying the project are smaller than the costs. The Classic NPV has the highest drop rate from all the options followed by the Forward Start NPV. The European-style Option NPV, the finite and the perpetual versions of American-style Option have a similar rate drop in absolute terms.

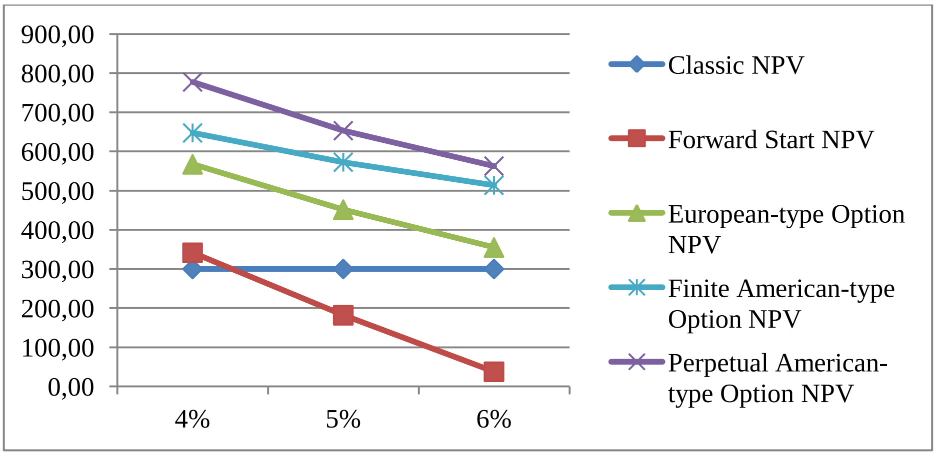
6.2.2 Dividend yields

From graph 3, we can conclude that a higher dividend yield always provides a destruction of value to the manager. The Classic NPV is not affected by the dividend yield because the parameter is not part of the calculation for Classic NPV since the decision of a manager confronted with a Classic NPV project and the execution of it can’t be postponed. All the alternative scenarios to the Classic NPV seem to have a similar drop rate in absolute terms except for the finite American-style Option whose drop rate is lower.

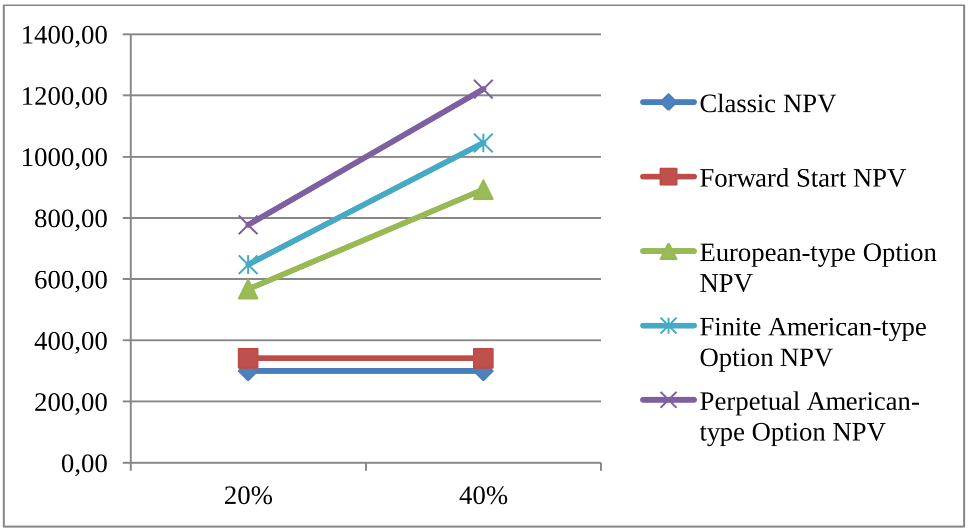
Graph 2 NPV for different Investment Costs



Graph 3 NPV for different dividend yields



Graph 4 NPV for different standard deviations



6.2.3 Uncertainty

From graph 4, it’s possible to see that the exposed variance of the project does not affect either the Classic NPV or the Forward Start NPV. This is explained by the fact that the decision to advance to the project is made immediately in this cases, and so, that decision isn’t exposed by the different market condition that can occur from that point to beyond. In the transition the a more risky situation form 20% of standard deviation to 40%, we can affirm that either the European-style Option NPV and both the finite and perpetual versions of the American-style Option gain value from a riskier situation, and that in relative terms have a similar gain in the 57% to 62% value gain order.

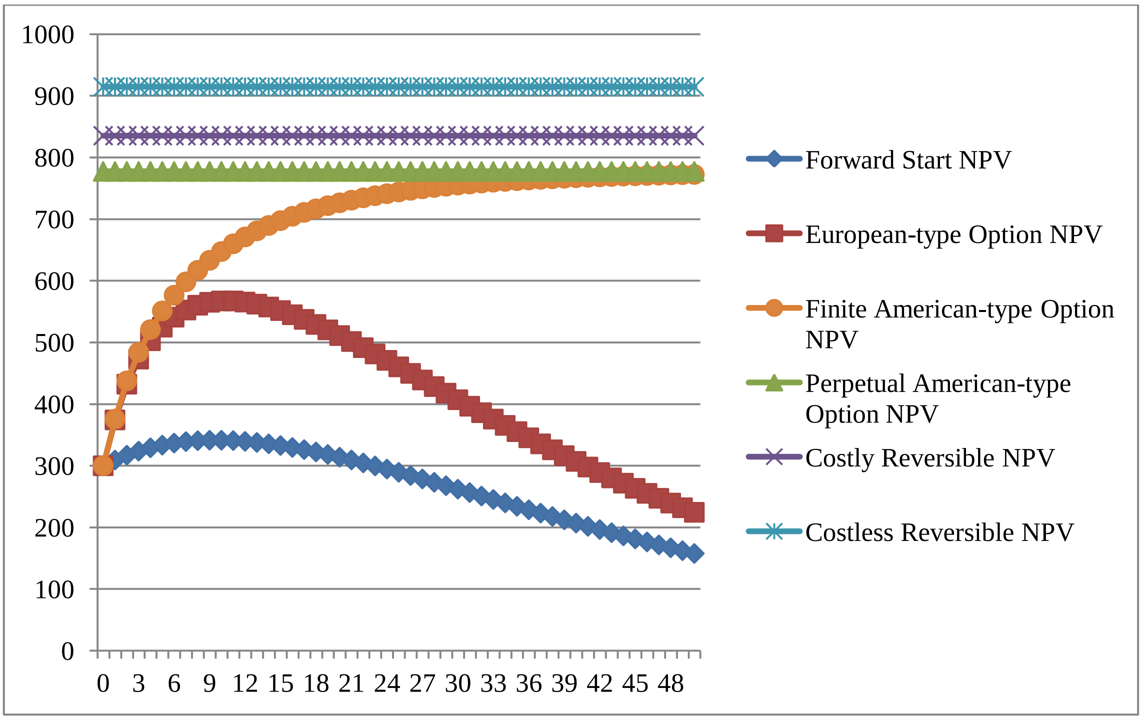
6.2.4 Time

From Graph 5, we can conclude that some types of NPV derived from the flexibility of the manager don’t depend of the time. The perpetual American-style NPV, the costly reversible NPV and the costless reversible NPV give to the manager the ability to postpone endlessly in time the decision to invest, so this kind of options are not affected by time.

In relation to the finite American-style option NPV, the value of this option grows with an augment of years. This makes sense because with more time, the manager has a bigger opportunity to reach the perfect time to invest in a project. The augment of years has enormous impact in the value of the option in the first 10 years, however, as we advance in time this augment in years turns in a residual gain in value reaching to the value of the perpetual American-style option NPV. That makes sense since the value of a finite American-style option NPV can’t be higher that is perpetual version because the perpetual version does not have a time horizon. As much as we advance in time, having no time horizon to decide is better for the manager that having a long one.

We can see that both the Forward Start NPV and the European-style option NPV have a period in time when the NPV rises, and then, the NPV decrease endlessly in time. In this case, the Forward Start NPV start to decay at year 11 and the European-style Option NPV start to decay at year 12. This shows that this kind of flexibility for these cases is only beneficial for a certain interval of time, and after the end of that time, the augment of years in the Forward Start NPV shows that the benefits of postponing start to be lower that the costs, and in the European-style Option NPV, the fact that the decision to invest is so delayed in time kills value to the manager. At year 23 for the Forward Start NPV, and at year 41 for the European Start NPV, the value of these options is lower than the Classic NPV, this means that from this years to beyond this kind of options don't give any value increase in relation to the Classic NPV.

Graph 5 NPV for different times till t=50



6.2.5 Thresholds

Thresholds are a very important factor in projects. When the value of a project meets the entry threshold of a project, is the point when the manager should invest in the project, and when the value meets the exit threshold, is the point when the manager should divest from the project.

In the creation process of some NPV's, it's necessary to determine the values of the open and exit thresholds in order to obtain the value of the NPV. For these cases it will be analyzed with the help of Table 3.

Table 3 Entry and Exit Thresholds value for different levels of flexibility

Option	$\overline{V}$	$\underline{V}$
Perpetual American-style Option NPV	4785,86	-
Costly Reversible NPV (Perpetual)	4048	1576
Costless Reversible NPV (Perpetual)	2750	2750

From the table we see that a higher level of flexibility corresponds to a lower entry threshold value and a higher exit threshold value. This is very simple to explain, all this options on table 3 give the manager the possibility to decide to go or not go to a project endlessly in time so, if the manager has more features of flexibility in the option like divest at any time if the project is no longer good, it is normal that the value of entry has to reflect this feature since once committed to the project the manager is exposed to less risk if he has the option to divest.

Table 4 Value, Entry and Exit Thresholds of the costly reversible NPV for different alpha parameters

α	$\underline{X}$	NPV	$\overline{V}$	$\underline{V}$
0,01	22	777,41	4785,85	12,7
0,25	550	779,36	4761,43	342,89
0,5	1100	788,62	4646,52	746,86
0,68	1500	803,94	4458,59	1102,53
0,86	1900	835,34	4077,7	1576,21
1	2200	914,58	2750	2750

6.2.6 Alphas

With the data of table 4 we can take some conclusions regarding the costless reversible perpetual option. As already stated, an α=1 is the case where the disinvestment benefit equals the investment cost, so the reversion in the decision to enter has no cost, corresponding to the costless reversible NPV. A higher alpha is a result of a higher disinvestment value (which correspond to a benefit), so it is natural that will result in a higher value of the option, the willing to an investor to entry the project is settled in lower values (entry threshold) and the willing to an investor to abandon the project is set at higher values since the investor can divest receiving a higher value.

It is important to analyze the case when α=0,01 (an α=0 it's impossible to compute with the equations considered in the *Mathematica 7.0* file). The values of NPV and entry threshold are virtually the same of the perpetual American-style Option. This is explained by the fact that an α=0 correspond to a 0 disinvestment value, that is, the reversion can’t add any value to the manager, so an α=0 will correspond to a perpetual American-style Option.

The evolution of the value of the option with respect to changes on the α parameter is not constant between the situation of α=0 (perpetual American-style version) and α=1 (costless reversible case). With a higher α, the augment of value of the option is always higher than the relative difference of the α.

7. Constant Elasticity of Variance diffusion case study

Under the Geometric Brownian Motion is a model which assumes constant volatility, so in order to compute the value of some assets or projects, the GBM can induce the manager into wrong decisions since a wide variety of assets and projects volatility do not have a constant behavior. The CEV model of Cox and Ross (1976) captures the leverage effect present in various assets and markets. To analyze this model and its implications for managers, the Dias and Nunes (2011) formulas for pricing perpetual American-style call and put option under the CEV diffusion are used in this chapter.

To accomplish this task, the same values of the base case used in the previous chapter (the values of Table 1) are considered as well as the values used for alternative scenarios. However, in this chapter we are also considering a β parameter. The elastic parameter is used to incorporate the so called leverage effect and volatility smile. The different β parameters considered for the case will be the same considered by Dias and Nunes (2011): 3, 2, 1, 0, -2, -4 and -6. As already stated, β = 2 is the case correspondent to the valuation with the Geometric Brownian Motion, and the key parameter of comparison.

7.1. Base case

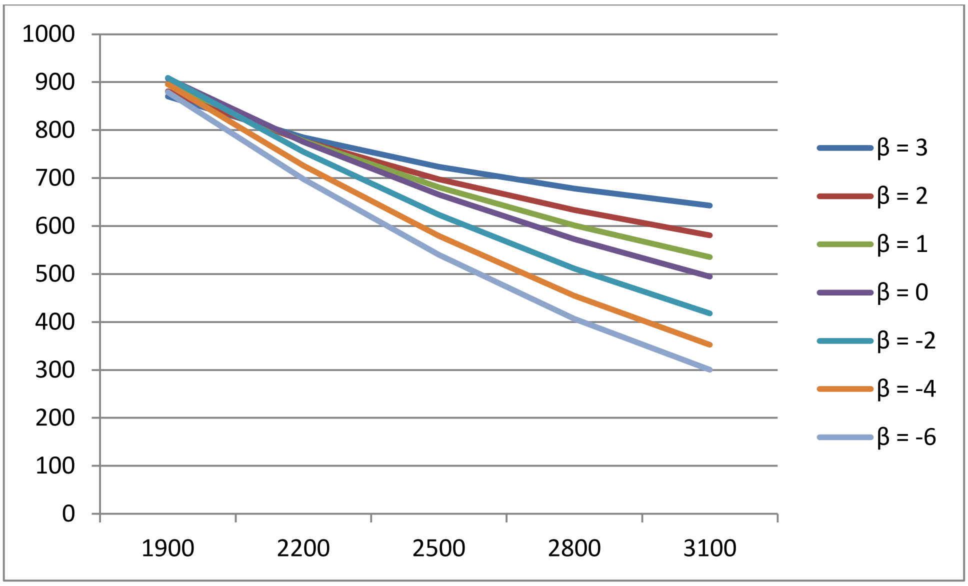
As the system of equations implied in the Dias and Nunes (2011) formula is too complex, the calculations of the NPV of the option and its respective threshold are calculated via *Mathematica 7.0*. The following values are obtained for the base case considering the various β parameters in Table 5 for the call option.

Table 5 Perpetual American-style Options NPV and threshold under the CEV diffusion model

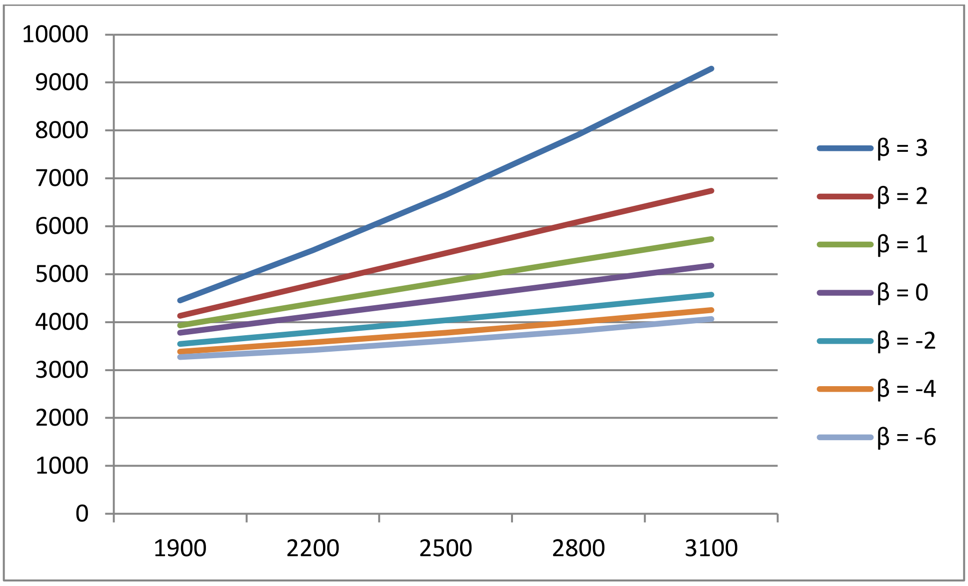
β	**3**	**2**	**1**	**0**	**-2**	**-4**	**-6**
NPV	784,54	777,41	777,23	775,47	754,94	726,11	697,81
$\bar{V}$	5499,75	4785,86	4392,42	4131,26	3790,61	3577,1	3420,53

Table 5 shows that a lower β of the project leads to a lower value for the option as well as to a lower entry threshold. This means that for the most extreme case (β = -6), if the manager assumes constant volatility (β = 2, the GBM case), the manager would incur in a error of evaluation of 10,24% in relation to the option NPV and an error of 28,53% relative to the entry threshold. For all the cases of β<2, also known in the CEV model as having a direct leverage effect, the manager, if assuming the GBM process, is waiting too much to invest in the project, thus being suboptimal.

Graph 6 Option value for different investment costs



Graph 7 Entry Threshold for different investment costs

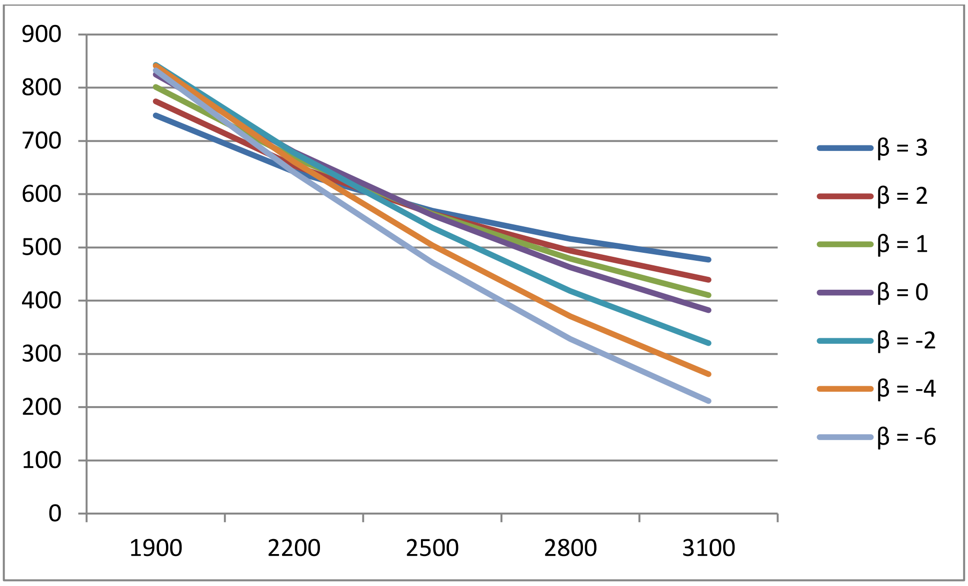


7.2. Alternative Scenarios

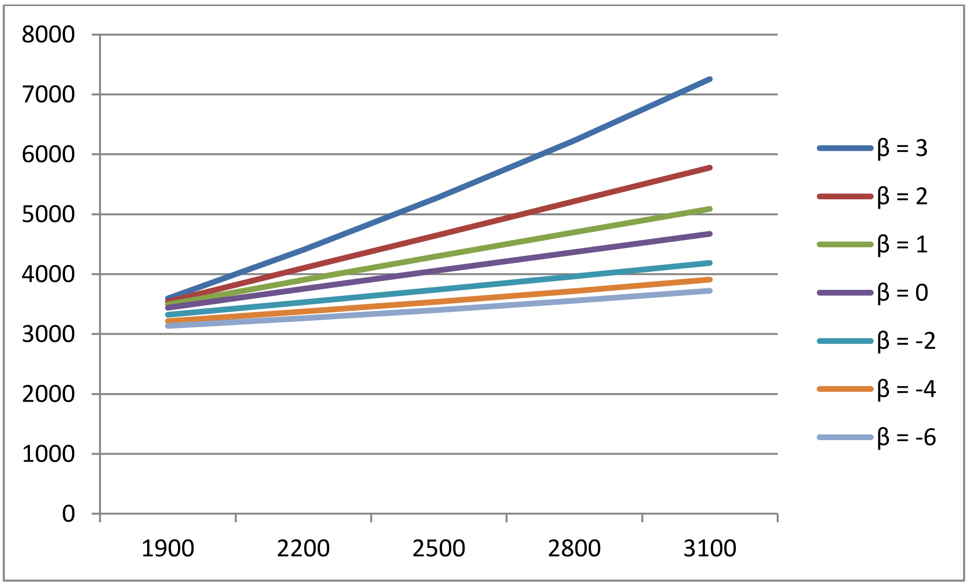
7.2.1. Investment Cost

Based on the insights of Graphs 6 and 7, we conclude that a higher β parameter will mean a higher value of entry threshold and as usually a higher investment cost will result into a higher investment threshold. The option value tends also to be higher with a higher β parameter. However that is not the case when the investment cost is 1900. When the investment cost is 1900, the option value from β=3 to β=-2, instead of decreasing, augments. This seems to provide evidence that when the investment cost goes way lower in relation to the project value (V_0), the tendency of a higher β meaning a higher option value can change.

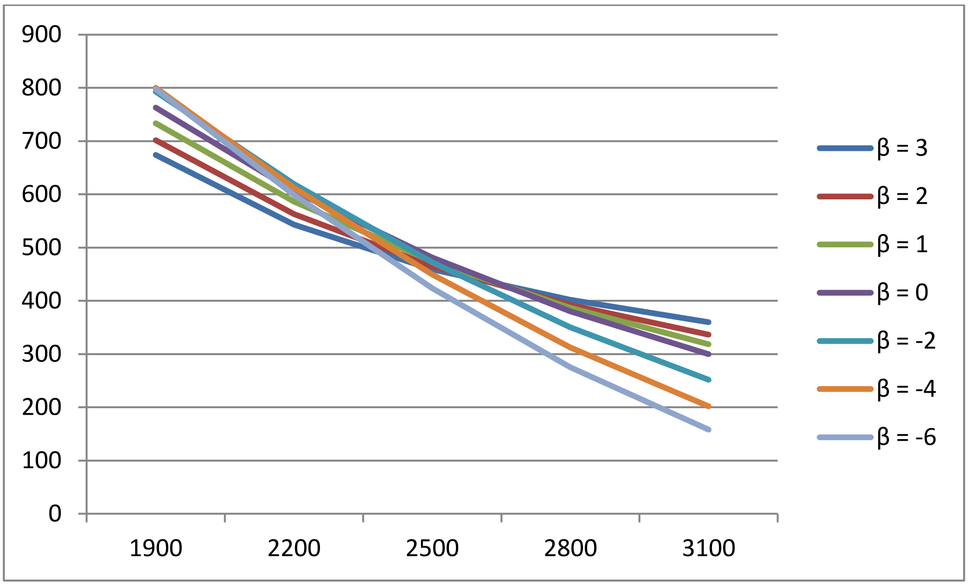
Graph 8 Option value for different investment costs with a 5% dividend yield



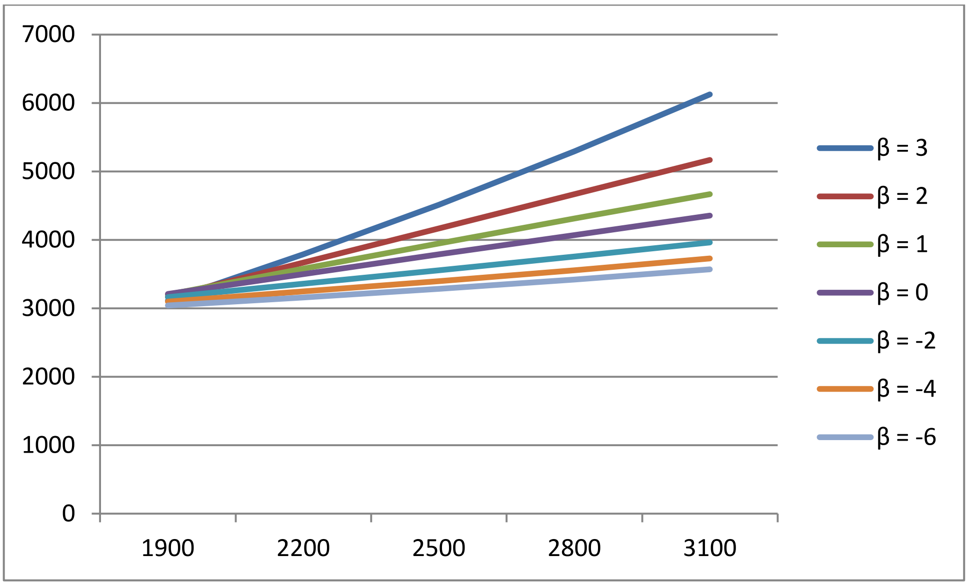
Graph 9 Entry Threshold for different investment costs with a 5% dividend yield



Graph 10 Option value for different investment costs with a 6% dividend yield



Graph 11 Entry Threshold for different investment costs with a 6% dividend yield

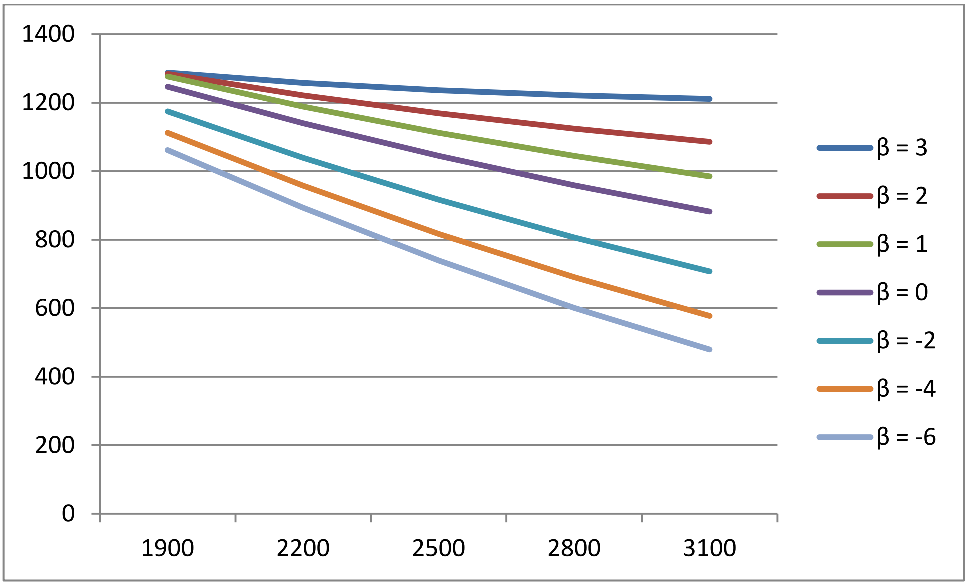


7.2.2. Dividend yield

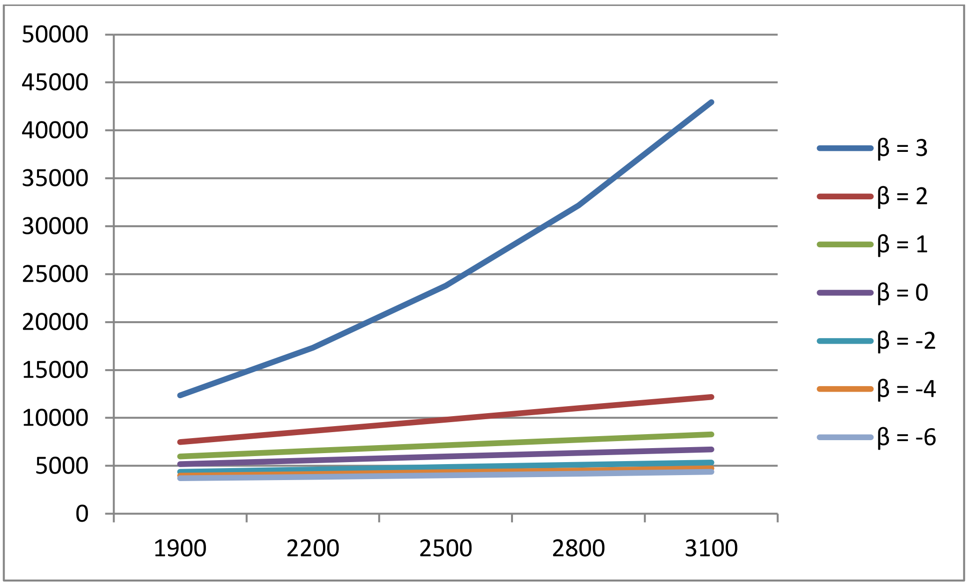
Graphs 8,9, 10 and 11 provide additional insights about the change in the dividend yield from 4% to 5%, in order to match the value of the risk-free rate, and to 6%, in order to surpass the risk-free rate. The entry thresholds seem to follow the general conclusions already taken about the β parameter, but we can conclude that a higher dividend yield will turn into a lower threshold and the higher the β, the higher the sensibility of the value change.

With respect to the option value, this change in volatility stresses the "anomaly" highlighted in Graph 6 for the values verified when a 1900 investment cost is considered. When the dividend yield is 5% and 6%, several β parameters are no longer positively correlated with the option value like it happens in the base case. Hence, it is possible to conclude that a combination of a higher dividend yield and lower investment cost describes a tendency where the β parameter and the respective option value turn to be negatively correlated.

Graph 12 Option value for different investment costs with a 40% standard deviation



Graph 13 Entry Threshold for different investment costs with a 40% standard deviation



7.2.3 Uncertainty

As concluded before in the GBM chapter, the augment of risk to a 40% standard deviation level from the 20% of the base case turns the value of the option higher as well as its entry threshold, and that expands to the CEV model in all β parameters as we can see from comparing graph 6 to graph 12 and graph 7 to graph 13. However, when β=3 (case of inverse leverage effect), the sensibility of this augment is higher as the investment cost of the option rises. In a direct leverage effect where β<2, all the augments seem constant with each other.

8. Conclusion

Regarding investment decisions, managers should not be limited by the traditional NPV taught in most business schools. An enormous number of factors can influence the way managers can get value from projects, and the way that value is calculated.

The classic NPV requires a decision to go with the project right away, as well as the investment made and there is no possible of reversion. However, as seen in this thesis, different kind of projects can delay the investment, delay the decision to invest into a certain finite or perpetual time, or can be reversed. Multiple ways of flexibility can drastically change the value that a manager gets from an investment project.

As table 2 on chapter 6 shows, the most flexible way of determining the project NPV value, the costless reversible NPV where a manager can delay the decision to invest endlessly in time, and once made the investment, can revert the decision with the disinvestment benefit equal to the investment cost, add in the thesis case 204,86% of the value of the Classic NPV. In the thesis base case, all the different kinds of flexibility add value to the classic NPV. When a manager delays the project to a pre-committed 10 year start date, the value for the manager in relation to the classic NPV rises 13,81%, when a manager can delay the decision of investment 10 years and only decide whether to invest or not after that 10 years, the NPV rises 89,1%, when a manager can decide to invest or not in the project during the next 10 years, the NPV rises 115,81%, when a manager can decide whether to invest or not in the project endlessly in time, the NPV rises 159,14%, and when the manager can decide whether to invest or not in the project endlessly in time, and after the investment is done, can revert the decision endlessly in time at a disinvestment benefit lower than the investment cost ($\alpha = 86\%$), the NPV rises 178,45%.

Besides the base case of this thesis points to the conclusion that flexibility in a project always add value to the project, this is not entirely correct when we make a sensitive analyzes to different parameters of the project such as the investment cost, the dividend yield and the time. Graph 5 shows that after a certain amount of time, the forward start NPV and the European-style NPV offer less value than the classic NPV, graph 2 shows that a lower investment cost can make the forward start NPV value less than the classic NPV as well as graph 3 shows the same for a higher dividend yield.

The calculations made on this thesis for the real options in chapter 6 are computed using a Geometric Brownian motion, which assumes constant volatility of the projects or assets. Chapter 7 of this thesis uses a different approach where volatility is not constant, capturing the leverage effect present in the different assets and markets, the Constant Elasticity of Variance diffusion.

Table 5 of chapter 7, compares the GBM with the CEV model for the perpetual American-style option, and clearly shows that for $\beta < 2$ (direct leverage effect), the CEV model tells the manager need to invest earlier (at a lower threshold) than the investment timing needed in the GBM model. In the uncommon case seen in markets of $\beta > 2$ (inverse leverage effect), the manager need to invest later than the investment timing needed in the GBM model. In both occasions, it is proven managers need to take in account the model in which he makers the real options valuations to avoid valuation errors, avoiding being suboptimal in their investment decisions. In the thesis case, if an asset follows a CEV diffusion process with $\beta = 6$, the manager would incur in a 28,53% error between the CEV and the GBM models, regarding the entry threshold, this is, when he invests, the manager with the GBM model should have invested when the present value of cash-flows reached a value 28,53% lower. Alternative scenarios provided by different values of investment costs, dividend yields and uncertainty have impact on the CEV valuation as seen in chapter 7.

When analyzing various models, managers also should take into account mean reverting process models, appropriated when an asset tends to a long run value. A triple comparison between the NPV values and thresholds of the Geometric Brownian Motion, the Constant Variance of Elasticity model and a mean reverting process would be interesting to study, however this study is left for future analysis.

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