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M|G|∞ QUEUE SYSTEM TRANSIENT BEHAVIOUR AND BUSY PERIOD

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Abstract. This paper main subject is the M|G|∞ queue system transient probabilities study as time functions. We achieve it completely when the time origin is an unoccupied system instant. But we do not get such a goal when the time origin is a busy period beginning instant. We shall see that, in this last situation, the service time length distribution hazard rate function plays a very important role. And so the results got may be useful in the survival analysis field. As the M|G|∞ queue system can be applied in the modelation of many social problems: sickness, unemployment, emigration, ...(see, for instance, Ferreira (1995 and 1996)), in these situations it is very important to study the busy period length distribution of that system. We show, in this work, that the solution of the problem may be in the resolution of a Ricatti equation generalizing the work of Ferreira (1998) as a consequence of the transient behaviour study.

Key words: M|G|∞, transient behaviour, hazard rate function, busy period, Ricatti equation

1 Introduction

We call M|G|∞ a queue system at which the customers arrive according to a Poisson process at rate λ, receive a service whose time length is a positive random variable with distribution function $G(.)$ and mean α. So $\alpha = \int_0^\infty [1-G(v)]dv$. Upon its arrival each customer finds immediately a server available. Each customer service is independent from the other customers services and from the arrival process. The traffic intensity is $\rho = \lambda\alpha$.

Being $N(t)$ the occupied servers number, that is the same that the being served customers number, at time t in a M|G|∞ system, according to Carrillo (1991) we have, putting $p_{0n}(t) = P[N(t) = n | N(0) = 0]$, $n = 0,1,2,...,$ that

$$p_{0n}(t)=\frac{\left(\lambda\int_0^t[1-G(v)]dv\right)^n}{n!}e^{-\lambda\int_0^t[1-G(v)]dv},n=0,1,2,... \qquad (1.1).$$

So, the transient distribution, being the time origin an empty system instant, is Poisson with mean $\lambda\int_0^t[1-G(v)]dv$.

The stationary distribution is the limit one:

$$\lim_{t\to\infty} p_{0n}(t)=\frac{\rho^n}{n!}e^{-\rho},n=0,1,2,...$$

being Poisson with mean ρ.

Be $p_{1'n}(t)=P[N(t)=n|N(0)=1'],n=0,1,2,...,$ meaning $N(0)=1'$ that the time origin is the one of a customer arrival at the system, making the number of served customers jump from 0 to 1. That is: a busy period begins.

At $t\geq 0$, possibly:

The customer that arrived at the time origin abandoned the system, with probability $G(t)$, or goes on being served, with probability $1-G(t)$;

The other servers, that were unoccupied at the time origin, are still unoccupied or occupied with $1,2,...$ customers, with probabilities given by $p_{0n}(t),n=1,2,...$

Both subsystems, the one of the initial customer and the other of the initially unoccupied servers, are independent and so we have

$$p_{1'0}(t)=p_{00}(t)G(t) \qquad (1.2)$$

$$p_{1'n}(t)=p_{0n}(t)G(t)+p_{0n-1}(t)(1-G(t)),n=1,2,...$$

For the M|M|∞ system (exponential service times) (1.2) is valid even if $N(0)=1$, that is: since the time origin in an instant at which there is one only customer in the system, simply, owing to the exponential distribution lack of memory. So

$$p_{10}^{M}(t)=\left(1-e^{-\frac{t}{\alpha}}\right)e^{-\rho\left(1-e^{-\frac{t}{\alpha}}\right)}$$

$$p_{1n}^{M}(t)=\frac{1}{(n-1)!}\rho^{n-1}\left(1-e^{-\frac{t}{\alpha}}\right)^{n-1}e^{-\rho\left(1-e^{-t/\alpha}\right)}\left(\frac{\left(1-e^{-\frac{t}{\alpha}}\right)^2\rho}{n}+e^{-\frac{t}{\alpha}}\right),n=1,2,...$$

It is easy to show that

$$\lim_{t\to\infty} p_{1'n}(t) = \frac{\rho^n}{n!}e^{-\rho}, n = 0,1,2,...$$

Calling $\mu(1',t)$ and $\mu(0,t)$ the mean values associated to the distributions given by (1.2) and (1.1), respectively, we have that

$$\mu(1',t) = \sum_{n=1}^{\infty} np_{1'n}(t) = \sum_{n=1}^{\infty} nG(t)p_{00}(t) + \sum_{n=1}^{\infty} np_{0n-1}(t)(1-G(t)) =$$

$$G(t)\mu(0,t) + (1-G(t))\sum_{j=0}^{\infty}(j+1)p_{0j}(t) = (0,t) + (1-G(t)).\ \text{So}$$

$$\mu(1',t) = 1 - G(t) + \lambda\int_0^t [1-G(v)]dv \tag{1.3}$$

We intend to present some results about $p_{0n}(t), n = 0,1,2,...,\ p_{1'0}(t)$ and $\mu(1',t)$ behaviours as time functions. We will show too that the $p_{1'0}(t)$ study induces a Ricatti equation important to the determination of a M | G | ∞ systems collection with practically exponential busy period.

2 $p_{0n}(t)$, $n = 0,1,2,...$ Behaviour as Time Function

This section main result is:

Proposition 2.1

If $G(t) < 1,\ t > 0$, continuous and differentiable

a) $p_{00}(t),\ t > 0$ is a decreasing function,

b) $p_{0n}(t),\ n \geq \rho,\ t > 0$ is an increasing function,

c) $p_{0n}(t),\ 0 < n < \rho,\ \rho > 1$

i) increases in $]0,t_n[$ being t_n given by

$$\int_0^{t_n} [1-G(v)]dv = \frac{n}{\lambda} \tag{2.1},$$

ii) decreases in $]t_n,\infty[$ and

iii) the $p_{0n}(t)$ maximum is

$$p_{0n}(t_n) = \frac{n^n}{n!}e^{-n} \tag{2.2}$$

Dem: a) is evident since

$$p_{00}(t) = e^{-\lambda\int_0^t [1-G(v)]dv}$$

For $n \geq 1\ \ \frac{d}{dt}p_{0n}(t) = \lambda p_{0n}(t)(1-G(t))\left(\frac{n}{\lambda\int_0^t [1-G(v)]dv} - 1\right), t > 0.$

As $\lambda\int_0^t [1-G(v)]dv < \rho$, if $n \geq \rho, \frac{d}{dt}p_{0n}(t) > 0, t > 0$ and we conclude b). If

$n < \rho, \frac{d}{dt} p_{0n}(t) = 0 \Leftrightarrow \int_0^t [1 - G(v)]dv = \frac{n}{\lambda}$ and we have c).

Notes:

- Although t_n, given by (2.1), depends on the arrival rate and on the service time length distribution, that does not happen with $p_{0n}(t_n)$ given by (2.2),
- For a certain arrival rate and service time length distribution we have, evidently,

$$t_{n+1} \geq t_n$$

and, as

$$\frac{p_{0n+1}(t_{n+1})}{p_{0n}(t_n)} = \frac{(n+1)^{n+1}}{(n+1)!} e^{-n-1} \frac{n!}{n^n} e^n = (n+1)\left(\frac{n+1}{n}\right)^n \frac{e^{-1}}{n+1} =$$

$$= \left(1 + \frac{1}{n}\right) e^{-1} \leq e e^{-1} = 1,\ p_{0n+1}(t_{n+1}) \leq p_{on}(t_n)$$

Under Proposition (2.1) conditions, but with $1 - G(t) = 0, t \geq t_l$, (1.1) becomes

$$p_{0n}(t) = \frac{\left(\lambda \int_0^t [1 - G(v)]dv\right)^n}{n!} e^{-\lambda \int_0^t [1-G(v)]dv}, t \leq t_l \quad \text{and} \quad p_{0n}(t) = \frac{\rho^n}{n!} e^{-\rho}, t > t_l,$$

$$, n = 0,1,2,...$$

and, so, the Proposition (2.1) conclusions are still valid, but the values $\frac{\rho^n}{n!} e^{-\rho}, n = 0,1,2,...$ occur after $t = t_l$. Evidently, $t_n < t_l, 0 < n < \rho, \rho > 1$.

3 $p_{1'0}(t)$ Behaviour as Time Function

For the $p_{1'n}(t), n = 0,1,2,...$ it is not possible to perform such a complete study as for the $p_{0n}(t), n = 0,1,2,....$ But the results for $p_{1'0}(t)$ are very interesting as we will see. Now the important result is

Preposition 3.1
If $G(t) < 1, t > 0$, continuous, differentiable and

$$h(t) \geq \lambda G(t), t > 0 \qquad (3.1),$$

being $h(t)$ the hazard rate function associated to $G(.)$, $p_{1'0}(t)$ is non-decreasing.

Dem: It is enough to note that, under these conditions,

$$\frac{d}{dt}p_{1'0}(t)=p_{00}(t)(1-G(t))\left(\frac{g(t)}{1-G(t)}-\lambda G(t)\right) \text{ where } g(t)=\frac{d}{dt}G(t) \text{ and } h(t)=\frac{g(t)}{1-G(t)}.$$

Notes:

Note that

$$h(t)\geq\lambda \tag{3.2}$$

is a sufficient condition for (3.1) and so if the rate at which the services end is greater or equal than the customers arrival rate $p_{1'0}(t)$ is non-decreasing,

- For the $M|M|\infty$ system (3.2) is equivalent to

$$\rho\leq 1$$

- Evidently these results may be useful in the survival analysis fields.

Putting

$$h(t)-\lambda G(t)=\beta,\beta\in IR$$

we have a Ricatti equation whose solution is (note that $G(t)=1,\ t\geq 0$ is a solution)

$$G(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda},t\geq 0,-\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \tag{3.3},$$

see Ferreira (1998). For a $M|G|\infty$ system with this service time length distribution

$$p_{1'0}(t)=e^{-\rho}-\frac{\left(1-e^{-\rho}\right)\beta}{\lambda}e^{-(\lambda+\beta)t},t\geq 0,-\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1}$$

Concretely

- $\beta=-\lambda$ we get

$$p_{1'0}(t)=1,t\geq 0.$$

In fact, in this situation, $G(t)=1,t\geq 0$. So $G(.)$ is degenerated at the origin. That is: every customer has null service time length. So the system is never occupied,

- $\beta=0$

$$p_{1'0}(t)=e^{-\rho},t\geq 0$$

and so $p_{1'0}(t),t\geq 0$ is constant,

- $\beta=\frac{\lambda}{e^{\rho}-1}$

$$p_{1'0}(t)=e^{-\rho}\left(1-e^{-(\lambda+\beta)t}\right),t\geq 0$$

With the service time length distribution given by (3.3), (1.1) becomes

$$p_{0n}(t)=\frac{\left(-\log\left(e^{-\rho}+\left(1-e^{-\rho}\right)e^{-(\lambda+\beta)t}\right)\right)^{n}}{n!}\left(e^{-\rho}+\left(1-e^{-\rho}\right)e^{-(\lambda+\beta)t}\right),$$

$$,t\geq 0,-\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1}.$$

Calling T the random variable associated to $G(.)$ given by (3.3) we have, see Ferreira (1998 a)

$$\frac{\left(1-e^{-\rho}\right)e^{-\rho}}{\lambda}\frac{n!}{(\lambda+\beta)^{n-1}}\leq E\left[T^{n}\right]\leq\frac{e^{\rho}-1}{\lambda}\frac{n!}{(\lambda+\beta)^{n-1}},$$

$$,-\lambda<\beta\leq\frac{\lambda}{e^{\rho}-1},n=1,2,... \tag{3.4}$$

Notes:

- The expression (3.4) giving bounds for $E\left[T^{n}\right],n=1,2,...$ proves its existence,
- For $n=1$ (3.4) is unuseful because we know that $E[T]=\alpha$. Curiously, the upper bound is $\frac{e^{\rho}-1}{\lambda}$, the M | G | ∞ system busy period mean value,
- For $\beta=-\lambda,E\left[T^{n}\right]=0,n=1,2,...$ evidently.

See however that (3.3) may take the form

$$G(t)=\frac{1+\frac{\beta}{\lambda}\left(1-e^{\rho}\right)e^{-(\lambda+\beta)t}}{1-\left(1-e^{\rho}\right)e^{-(\lambda+\beta)t}},t\geq 0,-\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1}$$

and, since $\rho<\log 2$,

$$G(t)=\left(1+\frac{\beta}{\lambda}\left(1-e^{\rho}\right)e^{-(\lambda+\beta)t}\right)\sum_{k=0}^{\infty}\left(1-e^{\rho}\right)^{k}e^{-k(\lambda+\beta)t},$$

$$,t\geq 0,-\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \tag{3.5}$$

After (3.5) we can easily compute the T Laplace transform for $\rho<\log 2$. And then get

$$E\left[T^{n}\right]=-\left(1+\frac{\beta}{\lambda}\right)n!\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{\left(k(\lambda+\beta)\right)^{n}},-\lambda<\beta\leq\frac{\lambda}{e^{\rho}-1},\rho<\log 2,n=1,2,...$$

Notes:

- $E[T] = -\left(1+\frac{\beta}{\lambda}\right)\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^k}{k(\lambda+\beta)} = -\frac{\lambda+\beta}{\lambda(\lambda+\beta)}.\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)}{k} =$

$$= \frac{1}{\lambda}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(e^{\rho}-1\right)^k}{k} = \frac{1}{\lambda}\log e^{\rho} = \frac{\rho}{\lambda} = \alpha$$

- For $n \geq 2$ we must truncate the infinite sum. Taking only M terms, to get an error lesser or equal than ε we must have simultaneously

$$.M > \frac{1}{\lambda+\beta} - 1$$

$$.M > \log_{\left(e^{\rho}-1\right)}\frac{e^{\rho}\lambda}{n!(\lambda+\beta)} - 1$$

4 A M | G | ∞ Systems Collection with Exponential Busy Period

Putting now $h(t) - \lambda G(\lambda) = \beta(t)$ ($\beta(.)$ is any time function) we get

$$\frac{dG(t)}{dt} = -\lambda G^2(t) - (\beta(t) - \lambda)G(t) + \beta(t)$$

that is a Ricatti equation about $G(.)$.

Solving it, after noting that $G(t) = 1, t \geq 0$ is a solution again, we get

$$G(t) = 1 - \frac{1}{\lambda}\frac{\left(1-e^{-\rho}\right)e^{-\lambda t - \int_0^t \beta(u)du}}{\int_0^{\infty} e^{-\lambda w - \int_0^w \beta(u)du}dw - \left(1-e^{-\rho}\right)\int_0^t e^{-\lambda w - \int_0^w \beta(u)du}dw},$$

$$,t \geq 0, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1} \qquad (4.1).$$

Putting (4.1) in

$$\overline{B}(s) = 1 + \lambda^{-1}\left(s - \frac{1}{\int_0^{\infty} e^{-st-\lambda\int_0^t [1-G(v)]dv}dt}\right)$$

that is the M | G | ∞ busy period length Laplace transform, see Stadje (1985), we get

$$\overline{B}(s)=\frac{1-(s+\lambda)(1-G(0))L\left[e^{-\lambda t-\int_0^t\beta(u)du}\right]}{1-\lambda(1-G(0))L\left[e^{-\lambda t-\int_0^t\beta(u)du}\right]},-\lambda\le\frac{\int_0^t\beta(u)du}{t}\le\frac{\lambda}{e^{\rho}-1} \tag{4.2}$$

where L means Laplace transform and

$$G(0)=\frac{\lambda\int_0^{\infty}e^{-\lambda w-\int_0^w\beta(u)du}dw+e^{-\rho}-1}{\lambda\int_0^{\infty}e^{-\lambda w-\int_0^w\beta(u)du}dw}\ .$$

After (4.2) we can compute $\frac{1}{s}\overline{B}(s)$ whose inversion gives

$$B(t)=\left(1-(1-G(0))\left(e^{-\lambda t-\int_0^t\beta(u)du}+\lambda\int_0^t e^{-\lambda w-\int_0^w\beta(u)du}dw\right)\right)*$$

$$*\sum_{n=0}^{\infty}\lambda^n(1-G(0))^n\left(e^{-\lambda t-\int_0^t\beta(u)du}\right)^{*n},-\lambda\le\frac{\int_0^t\beta(u)du}{t}\le\frac{\lambda}{e^{\rho}-1} \tag{4.3}$$

for the M|G|∞ busy period d.f., where $*$ is the convolution operator.
If $\beta(t)=\beta$ (constant), we get (3.3) and

$$B^{\beta}(t)=1-\frac{\lambda+\beta}{\lambda}\left(1-e^{-\rho}\right)e^{-e^{-\rho}(\lambda+\beta)t},t\ge 0,-\lambda\le\beta\le\frac{\lambda}{e^{\rho}-1}$$

So, if the service time d.f. is given by (3.3) the M|G|∞ busy period d.f. is the a mixture of a degenerate distribution at the origin and an exponential distribution.

Finally note that, for $\beta=\frac{\lambda}{e^{\rho}-1}$, $B^{\beta}(t)=1-e^{-\frac{\lambda}{e^{\rho}-1}t}$, $t\ge 0$ (purely exponential). And $B(t)$, given by (4.3) satisfies

$$B(t)\ge 1-e^{-\frac{\lambda}{e^{\rho}-1}t},\ t\ge 0,\ -\lambda\le\frac{\int_0^t\beta(u)du}{t}\le\frac{\lambda}{e^{\rho}-1}$$

5 $\mu(1't)$ Behaviour as Time Function

In this situation:

Proposition 5.1
If $G(t) < 1, t > 0$, continuous, differentiable and

$$h(t) \leq \lambda \tag{5.1}$$

$\mu(1', t)$ is non-decreasing.

Dem: After (1.3) we have $\frac{d}{dt}\mu(1', t) = (1 - G(t))(\lambda - h(t))$.

Notes:

- If the rate at which the services end is lesser or equal than the customers arrival rate $\mu(1', t)$ is non-decreasing,
- For the $M|M|\infty$ (5.1) is equivalent to

$$\rho \geq 1$$

- These results may be useful in the survival analysis field.

Putting

$$h(t) = \lambda$$

obviously we get

$$G(t) = 1 - e^{-\lambda t}, t \geq 0$$

and, for this $M|M|\infty$ system,

$$\mu(1, t) = 1$$

6 Concluding Remarks

In queues practical applications often it is used the populational process stationary distribution. This happens generally because the transient distribution is very complex and unuseful. And so the stationary distribution is used as a good transient one approximation. But in various situations this is not true. So it is necessary to know as well as possible the transient behaviour.

The $M|G|\infty$ systems transient behaviour, with an unoccupied system instant time origin, is very well known and not too complex. We deduced the time origin at the beginning of a busy period transient distribution.

We presented here a transient behaviour study, with some interesting results, for the $M|G|\infty$ systems with a lot of possible applications, namely in survival analysis.

It was done more exhaustively for the $p_{0n}(t)$ than for the $p_{1'n}(t), n = 0,1,2,\ldots$, but in the former situation everything is easier than in the other. But the $p_{1'0}(t)$ study leads to very interesting results even that they are looked only from the mathematical point of view. And, no less important, it allows through the resolution of a Ricatti equation the determination of a $M|G|\infty$ infinite systems collection with a very simple busy period distribution: a mixture of a degenerate distribution at the origin and an exponential distribution.

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