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Finite maturity caps and floors for exchange options on continuous flows.

João de Almeida Martins

Master's in mathematical finance

Supervisors:
PhD, José Carlos Dias, Full Professor,
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PhD, Fernando Correia da Silva, Integrated Researcher,
Business Research Unit (BRU-Iscte)

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Department of Finance

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"This result is too beautiful to be false; it is more important to have beauty in one's equations than to have them fit experiment."

-Paul Dirac (1963)

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Resumo

Esta tese oferece soluções novas para avaliar caps e floors de maturidades perpétuas e finitas em fluxos contínuos considerando receitas e custos estocásticos. Primeiramente, apresentamos o trabalho de Margrabe (1978) e McDonald & Siegel (1985) sobre opções de troca do tipo europeu, assumindo que ambos os ativos pagam dividendos, e de seguida, seguindo as ideias de Shackleton & Wojakowski (2007), ampliamos a literatura anterior sobre caps e floors ao fornecer novas fórmulas analíticas para avaliar caps e floors de maturidade finita que são contingentes a fluxos contínuos usando o método da decomposição temporal.

Abstract

This thesis offers novel analytical solutions for evaluating perpetual and finite maturity caps and floors on continuous flows where both the revenue and the cost are stochastic. We first present the work of Margrabe (1978) and McDonald & Siegel (1985) on European exchange options, assuming both assets pay dividends, and then, following the insights of Shackleton & Wojakowski (2007) we extend the previous literature on caps and floors arrangements by providing new analytical formulae for valuing finite maturity caps and floors that are contingent on continuous flows using the time decomposition method.

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CHAPTER 1

Introduction

In finance, a cap is a type of interest rate (or commodity) derivative in which the buyer receives payments at the end of each (predetermined) period when the interest rate exceeds the agreed strike price. An example of a cap would be an agreement to receive a payment at the end of every month when the Secured Overnight Financing Rate (SOFR) rate exceeds 3% over the next 3 years. Similarly, an interest rate floor is a derivative contract in which the buyer receives payments at the end of each period when the interest rate is below the agreed strike price. Some caps and floors offer a fixed payment, but some can offer the difference between the underlying rate (or commodity price) and the agreed strike price. These derivatives can be analysed as a series of Black & Scholes (1973) and Merton (1973) European call (or put options), known as caplets (or floorlets), that offer the difference between the underlying rate (or commodity cost) and the strike price at maturities corresponding to the end of each period where the cap (or floor) is in existence. Thus, a cap or floor can be seen as the sum of a series of caplets (or floorlets) over the horizon of the cap's (or floor's) maturity.

The interest rate and commodity literature has valued discrete caplets and floorlets using Black (1976) model and their summation. In the limit, as the time between maturities of each caplet (or floorlet) becomes increasingly small, caps and floors on continuous flows can be described. These instruments have been studied in the work of Shackleton & Wojakowski (2007) where each caplet (or floorlet) is a Black & Scholes (1973) and Merton (1973) European call (or put) option.

Other streams of the literature have also studied contingent flows. Within real options, to name a few, McDonald & Siegel (1985) and Dixit & Pindyck (1994), all motivate, discuss and value, the perpetual continuous cash flow from a project that captures the positive part of a stochastic net profit while avoiding losses.

Using terminology from the real options literature, we now treat the interest rate (or commodity price) as the revenue process S of a project that generates cash at an instantaneous rate of flow Sdt. The literature above, now describes a flexible project as a sum of the instantaneous net profit, $S-K$ (revenue S less cost K), where the project can, at anytime and without a cost, cease operation for as long as the net profit is negative, and at anytime and without a cost, resume operation when the net profit is positive. Over an infinite horizon, these papers evaluate continuous perpetual cash flows which are the sum of an infinite continuum of European style options (caplets or floorlets), while Shackleton & Wojakowski (2007), study the finite horizon case. Analytical solutions for (perpetual and finite) caps and floors on continuous exchange flows have recently been given in Dias, Nunes & Silva (2024). This thesis will offer analytical formulae for evaluating (perpetual and finite) caps and floors on continuous flows using the time decomposition technique of Shackleton & Wojakowski (2007), though we will consider not only the revenue process S (hereafter S_1) to be stochastic but also the cost process K (hereafter S_2) to be also stochastic by nature. This will imply that the continuous cash flows are now not a sum of an infinite continuum of European style options but a sum of an infinite continuum of European exchange options, which are studied in the works of Margrabe (1978) and McDonald & Siegel (1985).

The thesis is organized as follows. In chapter 2 we will layout the model presented by Margrabe (1978) and McDonald & Siegel (1985) for valuing European exchange options assuming that both risky assets continuously pay dividends. In chapter 3 we will derive the analytical formulae for evaluating (perpetual and finite) caps and floors on continuous flows. Finally, chapter 4 concludes.

CHAPTER 2

Exchange options on dividend paying assets

We begin our work by laying out the model to price a claim to exchange an asset S_1 for another asset S_2 at time T, where these assets pay dividends continuously over time and can be correlated. Margrabe (1978) valued the claim where the assets do not pay dividends and McDonald & Siegel (1985) valued the claim where assets pay dividends. We will then value the claim using the change of martingale measure approach and present a closed form solution.

2.1. Model specification under the physical measure

Following Margrabe (1978) and McDonald & Siegel (1985) models, we suppose two $\mathbb{P}$-measured standard Brownian processes $(W^{\mathbb{P}}_{1,t})_{t\geq 0}$ and $(W^{\mathbb{P}}_{2,t})_{t\geq 0}$, which are defined on a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$. We assume that the price of the risky assets S_1 follows a geometric Brownian motion, that is

$$\frac{dS_{1,t}}{S_{1,t}} = (\mu_1 - \delta_1)dt + \sigma_1 dW^{\mathbb{P}}_{1,t} \tag{2.1}$$

and the price of the risky asset S_2 also follows a geometric Brownian motion, that is

$$\frac{dS_{2,t}}{S_{2,t}} = (\mu_2 - \delta_2)dt + \sigma_2 dW^{\mathbb{P}}_{2,t}, \tag{2.2}$$

with covariance

$$cov\left(\frac{dS_{1,t}}{S_{1,t}}, \frac{dS_{2,t}}{S_{2,t}}\right) = \rho_{1,2}\sigma_1\sigma_2 dt, \tag{2.3}$$

where the constants μ_1 and μ_2 are the expected rates of return on the risky asset S_1 and S_2, δ_1 and δ_2 are the corresponding dividend yields, σ_1 and σ_2 are the corresponding standard deviations and $\rho_{1,2}$ is the correlation between S_1 and S_2.

2.2. Model specification under the risk-neutral measure

For pricing purposes, equations (2.1) and (2.2) must be rewritten under measure $\mathbb{Q}$, that is a martingale measure associated to the *numéraire* money-market account.

We define $\lambda_i = \frac{\mu_i - r}{\sigma_i}$, where r is the risk-free interest rate, λ_i is the risk-premium and σ_i the standard deviation of the asset S_i. Replacing this in equations (2.1) and (2.2) yields the risk-neutral processes for the risky assets prices:

$$\begin{aligned}\frac{dS_{i,t}}{S_{i,t}} &= (r - \delta_i)dt + \sigma_i(dW^{\mathbb{P}}_{i,t} + \lambda_i dt), \;\; i \in \{1,2\} \\ &= (r - \delta_i)dt + \sigma_i dW^{\mathbb{Q}}_{i,t}, \;\; i \in \{1,2\}.\end{aligned} \tag{2.4}$$

The Brownian processes $dW^{\mathbb{Q}}_{i,t} \equiv dW^{\mathbb{P}}_{i,t} + \lambda_i dt$ are the new $\mathbb{Q}$-measured Brownian motions (with the same standard filtration as $dW^{\mathbb{P}}_{i,t}$).

2.3. Orthogonalization of the pricing system

For our approach it is more convenient to work not with correlated Brownian motions $W^{\mathbb{Q}}_{1,t}$ and $W^{\mathbb{Q}}_{2,t}$, but rather with standard $\mathbb{Q}-$independent Brownian motions $Z^{\mathbb{Q}}_{1,t}$ and $Z^{\mathbb{Q}}_{2,t}$.

Given the correlation structure of equations (2.1) and (2.2), $W^{\mathbb{Q}}_{1,t}$ and $W^{\mathbb{Q}}_{2,t}$ can be rewritten as a linear combination of $Z^{\mathbb{Q}}_{1,t}$ and $Z^{\mathbb{Q}}_{2,t}$. We define:

$$dW^{\mathbb{Q}}_{1,t} = \rho_{1,2}\, dZ^{\mathbb{Q}}_{2,t} + \sqrt{1-\rho_{1,2}^2}\, dZ^{\mathbb{Q}}_{1,t} \tag{2.5}$$

$$dW^{\mathbb{Q}}_{2,t} = dZ^{\mathbb{Q}}_{2,t}. \tag{2.6}$$

Then $(W^{\mathbb{Q}}_{1,t})_{t\geq 0}$ and $(W^{\mathbb{Q}}_{2,t})_{t\geq 0}$ are a continuous martingale with $W^{\mathbb{Q}}_{1,0} = 0$ and $W^{\mathbb{Q}}_{2,0} = 0$ and

$$\begin{aligned}
dW^{\mathbb{Q}}_{1,t} dW^{\mathbb{Q}}_{1,t} &= \rho_{1,2}^2 dZ^{\mathbb{Q}}_{2,t} dZ^{\mathbb{Q}}_{2,t} + 2\rho_{1,2}\sqrt{1-\rho_{1,2}^2} dZ^{\mathbb{Q}}_{1,t} dZ^{\mathbb{Q}}_{2,t} + (1-\rho_{1,2}^2) dZ^{\mathbb{Q}}_{1,t} dZ^{\mathbb{Q}}_{1,t} \\
&= \rho_{1,2}^2\, dt + 0 + (1-\rho_{1,2}^2)\, dt \\
&= dt \\
dW^{\mathbb{Q}}_{2,t} dW^{\mathbb{Q}}_{2,t} &= dZ^{\mathbb{Q}}_{2,t} dZ^{\mathbb{Q}}_{2,t} \\
&= dt,
\end{aligned}$$

because according to the one-dimensional Lévy's theorem $(W^{\mathbb{Q}}_{1,t})_{t\geq 0}$ and $(W^{\mathbb{Q}}_{2,t})_{t\geq 0}$ are Brownian motions under $\mathbb{Q}$.

The Brownian motions $(W^{\mathbb{Q}}_{1,t})_{t\geq 0}$ and $(W^{\mathbb{Q}}_{2,t})_{t\geq 0}$ are correlated. According to Itô's product rule,

$$\begin{aligned}
d(W^{\mathbb{Q}}_{1,t} W^{\mathbb{Q}}_{2,t}) &= W^{\mathbb{Q}}_{1,t} dW^{\mathbb{Q}}_{2,t} + W^{\mathbb{Q}}_{2,t} dW^{\mathbb{Q}}_{1,t} + dW^{\mathbb{Q}}_{1,t} dW^{\mathbb{Q}}_{2,t} \\
&= W^{\mathbb{Q}}_{1,t} dW^{\mathbb{Q}}_{2,t} + W^{\mathbb{Q}}_{2,t} dW^{\mathbb{Q}}_{1,t} + \rho_{1,2} dt
\end{aligned}$$

Integrating, we obtain

$$W^{\mathbb{Q}}_{1,t} W^{\mathbb{Q}}_{2,t} = \int_0^t W^{\mathbb{Q}}_{1,t} dW^{\mathbb{Q}}_{2,t} + \int_0^t W^{\mathbb{Q}}_{2,t} dW^{\mathbb{Q}}_{1,t} + \rho_{1,2} t$$

By definition, the Itô integrals on the right-hand of the previous equation have expectation zero, so the covariance of $(W^{\mathbb{Q}}_{1,t})_{t\geq 0}$ and $(W^{\mathbb{Q}}_{2,t})_{t\geq 0}$ is

$$\mathbb{E}[W^{\mathbb{Q}}_{1,t} W^{\mathbb{Q}}_{2,t}] = \rho_{1,2} t.$$

Because both $(W^{\mathbb{Q}}_{1,t})_{t\geq 0}$ and $(W^{\mathbb{Q}}_{2,t})_{t\geq 0}$ have standard deviation $\sqrt{t}$, the constant $\rho_{1,2}$ is the correlation between $(W^{\mathbb{Q}}_{1,t})_{t\geq 0}$ and $(W^{\mathbb{Q}}_{2,t})_{t\geq 0}$.

Therefore, using equation (2.4), the model can be restated, under measure $\mathbb{Q}$, as

$$\begin{aligned}
\frac{dS_{1,t}}{S_{1,t}} &= (r-\delta_1)dt + \sigma_1 \left(\rho_{1,2}\, dZ^{\mathbb{Q}}_{2,t} + \sqrt{1-\rho_{1,2}^2}\, dZ^{\mathbb{Q}}_{1,t}\right) \\
\frac{dS_{2,t}}{S_{2,t}} &= (r-\delta_2)dt + \sigma_2\, dZ^{\mathbb{Q}}_{2,t}
\end{aligned} \tag{2.7}$$

2.4. European exchange options

We will now define a claim to exchange an asset S_1 for another S_2 at maturity T.

Definition 2.1. *The time-T value of a European-style exchange option on the asset S_1, with strike price S_2 and expiry at time T is:*

$$m_T\left(S_{1,T}, S_{2,T}, T\right) = max\left(S_{1,T} - S_{2,T}, 0\right). \tag{2.8}$$

We will price this option as the time-t expectation value of its discounted cash-flows:

$$m_t(S_{1,t}, S_{2,t}, T) = e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left[max\left(S_{1,T} - S_{2,T}, 0\right)|\mathcal{F}_t\right], \tag{2.9}$$

where the probability measure $\mathbb{Q}$ is the risk-neutral probability for the pricing problem and $\tau = T - t$.

We could analytically solve the expectation problem, but we will use a change of *numéraire* measure to simplify it.

2.4.1. Change of *numéraire* approach

Applying the logarithm transformation for $S_{2,t}$, under the risk-neutral probability measure $\mathbb{Q}$, results that:

$$\begin{aligned} S_{2,T} &= S_{2,t}\exp\left[\left(r - \delta_2 - \frac{\sigma_2^2}{2}\right)\tau + \sigma_2\int_t^T dZ_{2,u}^{\mathbb{Q}}\right] \\ &= S_{2,t}e^{(r-\delta_2)\tau}\exp\left[\left(-\frac{\sigma_2^2}{2}\right)\tau + \sigma_2\int_t^T dZ_{2,u}^{\mathbb{Q}}\right]. \end{aligned} \tag{2.10}$$

we now define the Radon-Nikodým derivative as

$$\frac{d\mathbb{Q}^{S_2}}{d\mathbb{Q}} := \exp\left(-\frac{\sigma_2^2}{2}\tau + \sigma_2\int_t^T dZ_{2,u}^{\mathbb{Q}}\right), \tag{2.11}$$

or in matrix form

$$\frac{d\mathbb{Q}^{S_2}}{d\mathbb{Q}} := \exp\left(-\frac{1}{2}\int_t^T ||\Theta||^2 du - \int_t^T \Theta\,.\,dZ_u^{\mathbb{Q}}\right), \tag{2.12}$$

where $Z_u^{\mathbb{Q}} = (Z_{1,u}^{\mathbb{Q}}, Z_{2,u}^{\mathbb{Q}})$ is a multidimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and $\Theta = (0, -\sigma_2)$.

We can verify the Novikov condition:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\frac{1}{2}\int_t^T ||\Theta||^2 du\right)\right] &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\frac{1}{2}\sigma_2^2(T-t)\right)\right] \\ &= \exp\left(\frac{1}{2}\sigma_2^2\tau\right) < \infty, \end{aligned}$$

so we can apply the two dimensional Girsanov's theorem to prove that the new measure $\mathbb{Q}^{S_2}$ is equivalent to $\mathbb{Q}$ and we can define the relation between $Z_{1,t}^{\mathbb{Q}}$ and $Z_{1,t}^{\mathbb{Q}^{S_2}}$ as well as between $Z_{2,t}^{\mathbb{Q}}$ and $Z_{2,t}^{\mathbb{Q}^{S_2}}$:

$$dZ_{1,t}^{\mathbb{Q}^{S_2}} = dZ_{1,t}^{\mathbb{Q}} \tag{2.13}$$

$$dZ_{2,t}^{\mathbb{Q}^{S_2}} = dZ_{2,t}^{\mathbb{Q}} - \sigma_2 dt. \tag{2.14}$$

The process $Z_t^{\mathbb{Q}^{S_2}} = (Z_{1,t}^{\mathbb{Q}^{S_2}}, dZ_{2,t}^{\mathbb{Q}^{S_2}})$ is a 2-dimensional Brownian motion under the new measure $\mathbb{Q}^{S_2}$, hence the component processes $Z_{i,t}^{\mathbb{Q}^{S_2}}$ are independent under $\mathbb{Q}^{S_2}$.

With this and equations (2.10) and (2.11) we can rewrite the pricing equation (2.9) under the new measure $\mathbb{Q}^{S_2}$:

$$\begin{aligned}
m_t(S_{1,t}, S_{2,t}, T) &= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[\max(S_{1,T} - S_{2,T}, 0)\right] \\
&= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[\max\left(S_{2,T}\left(\frac{S_{1,T}}{S_{2,T}} - 1, 0\right)\right)\right] \\
&= e^{-r\tau}S_{2,t}e^{(r-\delta_2)\tau}\mathbb{E}^{\mathbb{Q}}\left[\max\left(\left(\frac{S_{1,T}}{S_{2,T}} - 1, 0\right)\frac{d\mathbb{Q}^{S_2}}{d\mathbb{Q}}\right)\right] \\
&= S_2 e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[\max\left(C_T - 1, 0\right)\right].
\end{aligned} \tag{2.15}$$

We still need to find the distribution for the ratio process $C_t = \frac{S_{1,t}}{S_{2,t}}$ under $\mathbb{Q}^{S_2}$. For this, we begin by writing the equation for the ratio process C_t under the risk-neutral measure $\mathbb{Q}$. By Itô's product rule:

$$\begin{aligned}
dC_t &= \frac{\partial C_t}{\partial t}dt + \frac{\partial C_t}{\partial S_{1,t}}dS_{1,t} \\
&\quad + \frac{\partial C_t}{\partial S_{2,t}}dS_{2,t} + \frac{1}{2}\left[\frac{\partial^2 C_t}{\partial S_{1,t}^2}(dS_{1,t})^2 + 2\frac{\partial^2 C_t}{\partial S_{1,t}\partial S_{2,t}}dS_{1,t}dS_{2,t} + \frac{\partial^2 C_t}{\partial S_{2,t}^2}(dS_{2,t})^2\right] \\
&= 0 + \frac{1}{S_{2,t}}\left[(r-\delta_1)S_{1,t}dt + \sigma_1 S_{1,t}dW_{1,t}^{\mathbb{Q}}\right] - \frac{S_{1,t}}{S_{2,t}^2}\left[(r-\delta_2)S_{2,t}dt + \sigma_2 S_{2,t}dW_{2,t}^{\mathbb{Q}}\right] \\
&\quad + 0 + \frac{1}{2}\left[-\frac{2}{S_{2,t}^2}\left((r-\delta_1)S_{1,t}dt + \sigma_1 S_{1,t}dW_{1,t}^{\mathbb{Q}}\right)\left((r-\delta_2)S_{2,t}dt + \sigma_2 S_{2,t}dW_{2,t}^{\mathbb{Q}}\right)\right] \\
&\quad + \frac{1}{2}\left[\frac{2S_{1,t}}{S_{2,t}^3}\left((r-\delta_2)S_{2,t}dt + \sigma_2 S_{2,t}dW_{2,t}^{\mathbb{Q}}\right)^2\right] \\
&= C_t(r-\delta_1)dt + C_t\sigma_1 dW_{1,t}^{\mathbb{Q}} - C_t(r-\delta_2)dt - C_t\sigma_2 dW_{2,t}^{\mathbb{Q}} - C_t\sigma_1\sigma_2\rho_{1,2}dt + C_t\sigma_2^2 dt.
\end{aligned}$$

Hence, under the risk-neutral probability measure $\mathbb{Q}$, we have:

$$\frac{dC_t}{C_t} = \left(\delta_2 - \delta_1 + \sigma_2^2 - \sigma_1\sigma_2\rho_{1,2}\right)dt + \sigma_1 dW_{1,t}^{\mathbb{Q}} - \sigma_2 dW_{2,t}^{\mathbb{Q}}. \tag{2.16}$$

We can rewrite this equation under the new measure $\mathbb{Q}^{S_2}$. Using equations (2.5) and (2.6) we have that:

$$\begin{aligned}
\frac{dC_t}{C_t} &= \left(\delta_2 - \delta_1 + \sigma_2^2 - \sigma_1\sigma_2\rho_{1,2}\right)dt + \sigma_1 dW_{1,t}^{\mathbb{Q}} - \sigma_2 dW_{2,t}^{\mathbb{Q}} \\
&= \left(\delta_2 - \delta_1 + \sigma_2^2 - \sigma_1\sigma_2\rho_{1,2}\right)dt + \sigma_1(\rho_{1,2}dZ_{2,t}^{\mathbb{Q}} + \sqrt{1-\rho_{1,2}^2}dZ_{1,t}^{\mathbb{Q}}) - \sigma_2 dZ_{2,t}^{\mathbb{Q}}.
\end{aligned}$$

Using equations (2.13) and (2.14) we have that:

$$
\begin{aligned}
\frac{dC_t}{C_t} &= \left(\delta_2 - \delta_1 + \sigma_2^2 - \sigma_1\sigma_2\rho_{1,2}\right) dt \\
&\quad + \sigma_1(\rho_{1,2}(dZ_{2,t}^{\mathbb{Q}^{S_2}} + \sigma_2 dt) + \sqrt{1-\rho_{1,2}^2}dZ_{1,t}^{\mathbb{Q}^{S_2}}) - \sigma_2(dZ_{2,t}^{\mathbb{Q}^{S_2}} + \sigma_2 dt) \\
&= (\delta_2 - \delta_1)\, dt + (\rho_{1,2}\sigma_1 - \sigma_2)dZ_{2,t}^{\mathbb{Q}^{S_2}} + \sigma_1\sqrt{1-\rho_{1,2}^2}dZ_{1,t}^{\mathbb{Q}^{S_2}},
\end{aligned}
\tag{2.17}
$$

where $Z_{1,t}^{\mathbb{Q}^{S_2}}$ and $Z_{2,t}^{\mathbb{Q}^{S_2}}$ are independent under $\mathbb{Q}^{S_2}$. Moreover, we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{S_2}}\left[(\rho_{1,2}\sigma_1 - \sigma_2)dZ_{2,t}^{\mathbb{Q}^{S_2}} + \sigma_1\sqrt{1-\rho_{1,2}^2}dZ_{1,t}^{\mathbb{Q}^{S_2}}\right] &= (\rho_{1,2}\sigma_1 - \sigma_2)\mathbb{E}^{\mathbb{Q}^{S_2}}\left[dZ_{2,t}^{\mathbb{Q}^{S_2}}\right] \\
&\quad + \sigma_1\sqrt{1-\rho_{1,2}^2}\,\mathbb{E}^{\mathbb{Q}^{S_2}}\left[dZ_{1,t}^{\mathbb{Q}^{S_2}}\right] \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[(\rho_{1,2}\sigma_1 - \sigma_2)dZ_{2,t}^{\mathbb{Q}^{S_2}} + \sigma_1\sqrt{1-\rho_{1,2}^2}dZ_{1,t}^{\mathbb{Q}^{S_2}}\right] &= (\rho_{1,2}\sigma_1 - \sigma_2)^2\ Var\left[dZ_{2,t}^{\mathbb{Q}^{S_2}}\right] \\
&\quad + \left(\sigma_1\sqrt{1-\rho_{1,2}^2}\right)^2 Var\left[dZ_{1,t}^{\mathbb{Q}^{S_2}}\right] \\
&= \left(\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1^2\sigma_2^2\right) dt
\end{aligned}
$$

Therefore, as $((\rho_{1,2}\sigma_1 - \sigma_2)dZ_{2,t}^{\mathbb{Q}^{S_2}} + \sigma_1\sqrt{1-\rho_{1,2}^2}dZ_{1,t}^{\mathbb{Q}^{S_2}}) \sim \mathcal{N}(0, \sigma_c^2 dt)$, we can rewrite equation (2.17) as

$$
\frac{dC_t}{C_t} = (\delta_2 - \delta_1)\, dt + \sigma_c\, dW_{c,t}^{\mathbb{Q}^{S_2}}
\tag{2.18}
$$

where $\sigma_c = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1^2\sigma_2^2}$ and $W_{c,t}^{\mathbb{Q}^{S_2}} = \frac{(\rho_{1,2}\sigma_1 - \sigma_2)dZ_{2,t}^{\mathbb{Q}^{S_2}} + \sigma_1\sqrt{1-\rho_{1,2}^2}dZ_{1,t}^{\mathbb{Q}^{S_2}}}{\sigma_c}$ is a Brownian motion[1] under $\mathbb{Q}^{S_2}$.

2.4.2. Pricing formula

The next proposition shows, the closed-form solution of a European-style exchange option.

PROPOSITION 2.1. *The price of a European-style exchange option at time* $t < T$ *where* $\tau = T - t$, *is given by:*

$$
m_t\left(S_{1,t}, S_{2,t}, T\right) = S_{1,t}e^{-\delta_1\tau}\Phi(d_1(S_{1,t}, S_{2,t}, \tau)) - S_{2,t}e^{-\delta_2\tau}\Phi(d_0(S_{1,t}, S_{2,t}, \tau))
\tag{2.19}
$$

[1]This can be proven using Lévy's Theorem knowing that $dZ_{1,t}^{\mathbb{Q}^{S_2}}$ and $dZ_{2,t}^{\mathbb{Q}^{S_2}}$ are independent Brownian motions under $\mathbb{Q}^{S_2}$.

where:

$\tau = T - t\,,$

$d_0(S_{1,t}, S_{2,t}, \tau) = \dfrac{\log\left(\frac{S_{1,t}}{S_{2,t}}\right) + \left(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2}\right)\tau}{\sigma_c\sqrt{\tau}}\,,$

$d_1(S_{1,t}, S_{2,t}, \tau) = d_2(S_{1,t}, S_{2,t}, \tau) + \sigma_c\sqrt{\tau}\,,$

$\sigma_c = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}\,,$

and $\Phi(*)$ *is the cumulative standard normal distribution.*

Proof.

$$\begin{aligned} m_t(S_{1,t}, S_{2,t}, T) &= S_2 e^{-\delta_2\tau}\mathbb{E}_t^{\mathbb{Q}^{S_2}}\left[\max\left(C_T - 1, 0\right)|\mathcal{F}_t\right] \\ &= S_2 e^{-\delta_2\tau}\mathbb{E}_t^{\mathbb{Q}^{S_2}}\left[\left(C_T - 1\right)\mathbb{1}_{\{C_T \geq 1\}}|\mathcal{F}_t\right]. \end{aligned} \tag{2.20}$$

We may write

$$\begin{aligned} C_T &= C_t \exp\left[(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2})(T-t) + \sigma_c\left(W_{c,T}^{\mathbb{Q}^{S_2}} - W_{c,t}^{\mathbb{Q}^{S_2}}\right)\right] \\ &= C_t \exp\left[(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2})\tau + \sigma_c\frac{W_{c,T}^{\mathbb{Q}^{S_2}} - W_{c,t}^{\mathbb{Q}^{S_2}}}{\sqrt{\tau}}\sqrt{\tau}\right] \\ &= C_t \exp\left[(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2})\tau - \sigma_c Y\sqrt{\tau}\right] \end{aligned} \tag{2.21}$$

where $Y = -\frac{W_{c,T}^{\mathbb{Q}^{S_2}} - W_{c,t}^{\mathbb{Q}^{S_2}}}{\sqrt{\tau}}$ is a standard normal random variable independent of $\mathcal{F}_t$ since $W_{c,T}^{\mathbb{Q}^{S_2}} - W_{c,t}^{\mathbb{Q}^{S_2}}$ is independent of $\mathcal{F}_t$.

Therefore, with $C_t = x$, equation (2.20) can be rewritten as

$$\begin{aligned} m_t(S_{1,t}, S_{2,t}, T) &= S_2 e^{-\delta_2\tau}\mathbb{E}_t^{\mathbb{Q}^{S_2}}\left[\left(C_T - 1\right)\mathbb{1}_{\{C_T \geq 1\}}|\mathcal{F}_t\right] \\ &= S_2 e^{-\delta_2\tau}\mathbb{E}_t^{\mathbb{Q}^{S_2}}\left[\left(x e^{(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2})\tau - \sigma_c Y\sqrt{\tau}} - 1\right)\mathbb{1}_{\{C_T \geq 1\}}|\mathcal{F}_t\right] \\ &= S_2 e^{-\delta_2\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(x e^{(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2})\tau - \sigma_c y\sqrt{\tau}} - 1\right)^+ e^{-\frac{1}{2}y^2}dy. \end{aligned}$$

The integrand

$$\left(x e^{(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2})\tau - \sigma_c y\sqrt{\tau}} - 1\right)^+$$

is positive if and only if

$$y \leq d_0(x, 1, \tau) = \frac{1}{\sigma_c\sqrt{\tau}}\left[\ln\left(\frac{x}{1}\right) + \left(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2}\right)\tau\right]. \tag{2.22}$$

Therefore,

$$
\begin{aligned}
m_t(S_{1,t}, S_{2,t}, T) &= S_2 e^{-\delta_2\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)}\left(xe^{(\delta_2-\delta_1-\frac{\sigma_c^2}{2})\tau-\sigma_c y\sqrt{\tau}}-1\right)e^{-\frac{1}{2}y^2}dy \\
&= S_2 e^{-\delta_2\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)}\left(xe^{(\delta_2-\delta_1-\frac{\sigma_c^2}{2})\tau-\sigma_c y\sqrt{\tau}}\right)e^{-\frac{1}{2}y^2}dy \\
&- S_2 e^{-\delta_2\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)} e^{-\frac{1}{2}y^2}dy \\
&= S_2 e^{-\delta_2\tau} x e^{(\delta_2-\delta_1)\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)} e^{-\frac{1}{2}(\sigma_c^2\tau+2\sigma_c y\sqrt{\tau}+y^2)}dy \\
&- S_2 e^{-\delta_2\tau}\Phi(d_0(x,1,\tau)) \\
&= S_2 e^{-\delta_2\tau} x e^{(\delta_2-\delta_1)\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)} e^{-\frac{1}{2}(y+\sigma_c\sqrt{\tau})^2}dy \\
&- S_2 e^{-\delta_2\tau}\Phi(d_0(x,1,\tau)).
\end{aligned}
$$

With the change of variable $z = y + \sigma_c\sqrt{\tau}$, where $d_1(x,\tau) = d_0(x,1,\tau) + \sigma_c\sqrt{\tau}$, then

$$
\begin{aligned}
m_t(S_{1,t}, S_{2,t}, T) &= S_2 e^{-\delta_2\tau} x e^{(\delta_2-\delta_1)\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)+\sigma_c\sqrt{\tau}} e^{-\frac{1}{2}(z)^2}dy \\
&- S_2 e^{-\delta_2\tau}\Phi(d_0(x,1,\tau)) \\
&= S_2 e^{-\delta_2\tau} x e^{(\delta_2-\delta_1)\tau}\Phi(d_1(x,1,\tau)) - S_2 e^{-\delta_2\tau}\Phi(d_0(x,1,\tau)).
\end{aligned}
$$

Finally, since $x = C_t = \frac{S_{1,t}}{S_{2,t}}$

$$
\begin{aligned}
m_t(S_{1,t}, S_{2,t}, T) &= S_{2,t} e^{-\delta_2\tau} x e^{(\delta_2-\delta_1)\tau}\Phi(d_1(x,1,\tau)) - S_{2,t} e^{-\delta_2\tau}\Phi(d_0(x,1,\tau)) \\
&= S_{1,t} e^{-\delta_1\tau}\Phi(d_1(S_{1,t}, S_{2,t},\tau)) - S_{2,t} e^{-\delta_2\tau}\Phi(d_0(S_{1,t}, S_{2,t},\tau)).
\end{aligned}
\tag{2.23}
$$

□

CHAPTER 3

Finite maturity caps for exchange options

We will now define the real options problem of this thesis, the model assumption and then derive the formulae for the perpetual case, the forward starting perpetual case and finally combine the two to get the finite case.

3.1. Real options problem

Following Dixit & Pindyck (1994, pages 186-195) and Shackleton & Wojakowski (2007), we consider a continuous price or revenue process S_1 that generates cash at an instantaneous rate of flow $S_1 dt$. The risk-neutral dynamics are described as:

$$\frac{dS_{1,t}}{S_{1,t}} = (r - \delta_1)dt + \sigma_1 dW^{\mathbb{Q}}_{1,t}, \tag{3.1}$$

where the constant r is the risk free interest rate, δ_1 is the dividend yields of asset S_1 and σ_1 is the corresponding standard deviation and $(W^{\mathbb{Q}}_{1,t})_{t\geq 0}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. We will also assume that operation of the project entails a flow cost S_2, but that the operation can be temporarily and costlessly suspended when S_1 falls below S_2, and costlessly resumed later if S_1 rises above S_2. In Dixit & Pindyck (1994, pages 186-195) and Shackleton & Wojakowski (2007) the flow cost S_2 is constant and known. For our work, we will consider that it follows the following risk-neutral dynamics:

$$\frac{dS_{2,t}}{S_{2,t}} = (r - \delta_2)dt + \sigma_2 dW^{\mathbb{Q}}_{2,t}, \tag{3.2}$$

where the constant r is the risk free interest rate, δ_2 is the dividend yields of asset S_2 and σ_2 is the corresponding standard deviation and $(W^{\mathbb{Q}}_{2,t})_{t\geq 0}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. We also assume that S_1 and S_2 can be correlated, that is

$$cov\left(\frac{dS_{1,t}}{S_{1,t}}, \frac{dS_{2,t}}{S_{2,t}}\right) = \rho_{1,2}\sigma_1\sigma_2 dt. \tag{3.3}$$

Therefore, at any instant t, the profit flow from this project is given by:

$$\pi(S_{1,t}, S_{2,t}) = \max\left[S_{1,t} - S_{2,t}, 0\right]. \tag{3.4}$$

Let $V(S_1, S_2, T)$ be the time-t value of a flexible finitely lived project (or profit spread cap). This value can be derived from a perpetual cash flow based on the maximum instantaneous flow rate, $\pi(S_{1,t}, S_{2,t})dt$. Under the risk-neutral measure, this is given by:

$$\begin{aligned} V_t(S_{1,t}, S_{2,t}, T) &= \int_t^T e^{-r(u-t)} \mathbb{E}^{\mathbb{Q}} \left[\pi(S_{1,u}, S_{2,u}) | \mathcal{F}_t \right] du \\ &= \int_t^T m_t(S_{1,u}\,, S_{2,u}\,, u) du, \end{aligned} \tag{3.5}$$

where $m_t(S_{1,t}\,, S_{2,t}\,, u)$ is interpreted as the time-t price of a exchange option (or caplet) on an asset S_1 with exercise price of another asset S_2, and with expiry date at time $u(\geq t)$, with fair price derived in chapter 2.

Even though it is possible to solve the integral (3.5) numerically, we will follow the insight of Shackleton and Wojakowski (2007) to evaluate this problem using the time decomposition method:

$$V_t(S_{1,t}, S_{2,t}, T) = V_t(S_{1,t}, S_{2,t}, \infty) - e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[V_T(S_{1,T}, S_{2,T}, \infty) | \mathcal{F}_t \right]. \tag{3.6}$$

To accomplish this, we will first derive the perpetual real options solution of this problem in the Section 3.2 and then the corresponding forward start perpetuity will be derived in the Section 3.3. Finally, in the Section 3.4, we will combine the two solutions to provide a formula for the finite problem above.

3.2. Perpetual case

3.2.1. Perpetual real options problem

The time-t value of an infinitely lived project or profit spread cap, $V(S_1, S_2, \infty)$, is the "continuous and perpetual sum" of instantaneous maximum flows, $\pi(S_{1,t}, S_{2,t})dt$, where $\pi(S_{1,t}, S_{2,t}) = \max(S_{1,t} - S_{2,t}, 0)$, over a stochastic revenue $S_{1,t}$ and a stochastic cost $S_{2,t}$. Its expected value computed under the risk-neutral measure $\mathbb{Q}$, and conditional to the information available until time t, is given by:

$$\begin{aligned} V_t(S_{1,t}, S_{2,t}, \infty) &= \int_t^\infty e^{-r(u-t)} \mathbb{E}^{\mathbb{Q}} \left[\pi(S_{1,u}, S_{2,u}) | \mathcal{F}_t \right] du \\ &= \int_t^\infty m_t(S_{1,t}, S_{2,t}, u) du. \end{aligned} \tag{3.7}$$

The perpetual real options problem under analysis needs to be specialized in each of the cases $S_{1,t} < S_{2,t}$ and $S_{1,t} \geq S_{2,t}$. For the range of prices where $S_{1,t} < S_{2,t}$ the (perpetual) claim $V(S_{1,t}, S_{2,t}, \infty)$ must satisfy the following partial differential equation (PDE):

$$\begin{aligned} &\frac{1}{2}\left(\sigma_1^2 S_{1,t}^2 \frac{\partial^2 V_t}{\partial S_{1,t}^2} + 2\rho_{1,2}\sigma_1\sigma_2 S_{1,t} S_{2,t} + \sigma_2^2 S_{2,t}^2 \frac{\partial^2 V_t}{\partial S_{2,t}^2} \right) + \\ &\qquad + (r - \delta_1)\, S_{1,t} \frac{\partial V_t}{\partial S_{1,t}} + (r - \delta_2)\, S_{2,t} \frac{\partial V_t}{\partial S_{2,t}} - rV_t = 0. \end{aligned} \tag{3.8}$$

For the range of prices where $S_{1,t} \geq S_{2,t}$ the (perpetual) claim $V_t(S_{1,t}, S_{2,t}, \infty)$ must satisfy a similar PDE, but now containing the profit flow $(S_{1,t} - S_{2,t})$, that is:

$$\begin{aligned}\frac{1}{2}\left(\sigma_1^2 S_{1,t}^2 \frac{\partial^2 V_t}{\partial S_{1,t}^2} + 2\rho_{1,2}\sigma_1\sigma_2 S_{1,t}S_{2,t} + \sigma_2^2 S_{2,t}^2 \frac{\partial^2 V_t}{\partial S_{2,t}^2}\right) + \\ + (r-\delta_1)\, S_{1,t}\frac{\partial V_t}{\partial S_{1,t}} + (r-\delta_2)\, S_{2,t}\frac{\partial V_t}{\partial S_{2,t}} - rV_t + (S_{1,t} - S_{2,t}) = 0.\end{aligned} \tag{3.9}$$

We can solve these PDEs in each of the ranges, $S_{1,t} < S_{2,t}$ and $S_{1,t} \geq S_{2,t}$ under some conditions and then "stitch them together" using the respective value matching and smooth pasting conditions.

3.2.2. Perpetual caps

Next proposition produces a formula for valuing the perpetual claim using the usual PDE theory with the appropriate constraints.

Proposition 3.1. *The time-t value of a perpetual cap on continuous flows is given by:*

$$V(S_{1,t}, S_{2,t}, \infty) = \begin{cases} \frac{1}{\beta_- - \beta_+}\left[\frac{\beta_- - 1}{\delta_1} - \frac{\beta_-}{\delta_2}\right] S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_+} & , S_{1,t} < S_{2,t} \\ \frac{1}{\beta_- - \beta_+}\left[\frac{\beta_+ - 1}{\delta_1} - \frac{\beta_+}{\delta_2}\right] S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_-} + \frac{S_{1,t}}{\delta_1} - \frac{S_{2,t}}{\delta_2} & , S_{1,t} \geq S_{2,t} \end{cases}, \tag{3.10}$$

where the constants, $\beta_+ > 1$ and $\beta_- < 0$, are given by:

$$\beta_\pm = \left(\frac{1}{2} - \frac{\delta_2 - \delta_1}{\sigma_c^2}\right) \pm \sqrt{\left(\frac{\delta_2 - \delta_1}{\sigma_c^2} - \frac{1}{2}\right)^2 + \frac{2\delta_2}{\sigma_c^2}}. \tag{3.11}$$

Proof. We let $V_t(x,y) = \lim_{\tau\to\infty} V_t(x,y,\tau)$ denote the value of the perpetual cap at time t, if the underlying assets at that time are $x = S_{1,t}$ and $y = S_{2,t}$, and that this (perpetual) cap follows the following PDE:

$$\begin{aligned}\frac{1}{2}\left(\sigma_1^2 x^2 \frac{\partial^2 V_t}{\partial x^2} + 2\rho_{1,2}\sigma_1\sigma_2 xy + \sigma_2^2 y^2 \frac{\partial^2 V_t}{\partial y^2}\right) + \\ + (r-\delta_1)\, x\frac{\partial V_t}{\partial x} + (r-\delta_2)\, y\frac{\partial V_t}{\partial y} - rV_t + \max(x-y, 0) = 0.\end{aligned} \tag{3.12}$$

We first note that, as stated by Margrabe (1978), the pricing formula of an option to exchange an asset for another, m_t, is homogeneous of degree 1 in x and y and therefore $V(x,y)$ is also homogeneous of degree 1. Hence, we can define $V_t(x,y) = y f_t(c)$ where $c =$

$\frac{x}{y}$ to simplify the problem to a 1-dimensional case. Successive differentiation yields:

$$\begin{aligned}
\frac{\partial V_t(x,y)}{\partial x} &= \frac{df_t(c)}{dc}, \\
\frac{\partial V_t(x,y)}{\partial y} &= f_t(c) - \frac{df_t(c)}{dc}\cdot c, \\
\frac{\partial^2 V_t(x,y)}{\partial x^2} &= \frac{d^2 f_t(c)}{dc^2}\cdot\frac{1}{y}, \\
\frac{\partial^2 V_t(x,y)}{\partial x \partial y} &= \frac{\partial^2 V_t(x,y)}{\partial y \partial x} = -\frac{d^2 f_t(c)}{dc^2}\cdot\frac{c}{y}, \\
\frac{\partial^2 V(x,y,t)}{\partial^2 y} &= \frac{d^2 f_t(c)}{dc^2}\cdot\frac{c^2}{y}.
\end{aligned}$$

Substituting this in the PDE (3.12) yields:

$$\begin{aligned}
&\frac{1}{2}\left[\sigma_1^2 x^2\left(\frac{d^2 f_t}{dc^2}\cdot\frac{1}{y}\right) + 2\rho_{1,2}\sigma_1\sigma_2 xy\left(-\frac{d^2 f_t}{dc^2}\cdot\frac{c}{y}\right) + \sigma_2^2 y^2\left(\frac{d^2 f_t}{dc^2}\cdot\frac{c^2}{y}\right)\right] \\
&+\left[(r-\delta_1)x\left(\frac{df_t}{dc}\right) + (r-\delta_2)y\left(f_t - c\frac{df_t}{dc}\right) - r\,(yf_t) + y\max(c-1,0)\right] = 0.
\end{aligned}$$

Dividing both sides by y (> 0):

$$\begin{aligned}
&\frac{1}{2}\left[\sigma_1^2 c^2\left(\frac{d^2 f_t}{dc^2}\right) + 2\rho_{1,2}\sigma_1\sigma_2 c^2\left(-\frac{d^2 f_t}{dc^2}\right) + \sigma_2^2 c^2\left(\frac{d^2 f_t}{dc^2}\right)\right] \\
&+\left[(r-\delta_1)c\left(\frac{df_t}{dc}\right) + (r-\delta_2)\left(f_t - c\frac{df_t}{dc}\right) - rf_t + \max(c-1,0)\right] = 0.
\end{aligned}$$

Simplifying the equation and grouping the terms yields:

$$\begin{aligned}
&\frac{1}{2}\left[\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2\right]c^2\frac{d^2 f_t}{dc^2} \\
&+\left[(\delta_2-\delta_1)c\frac{df_t}{dc} - \delta_2 f_t + \max(c-1,0)\right] = 0.
\end{aligned}$$

Defining $\sigma_c^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2$ we have:

$$\frac{\sigma_c^2 c^2}{2}\cdot\frac{d^2 f_t}{dc^2} + (\delta_2-\delta_1)c\frac{df_t}{dc} - \delta_2 f_t + \max(c-1,0) = 0. \tag{3.13}$$

We note that the homogeneous part of the ordinary diferential equation (ODE) (3.13) is one of Euler type:

$$\frac{\sigma_c^2 c^2}{2}\cdot\frac{d^2 f_t}{dc^2} + (\delta_2-\delta_1)c\frac{df_t}{dc} - \delta_2 f_t = 0, \tag{3.14}$$

which by the usual ODE theory has solutions of the form $f_t(c) = c^\beta$, where β is a constant to be determined. Substituting this solution into the equation (3.14) we see that the solution satisfies the equation provided β is the root of the quadratic equation, $q(\beta) = 0$, with $q(\beta)$:

$$q(\beta) := \frac{1}{2}\sigma^2\beta(\beta-1) + (\delta_2-\delta_1)\beta - \delta_2. \tag{3.15}$$

The equation above has two roots, β_+ (> 1) and β_- (< 0) given by:

$$\beta_\pm = \left(\frac{1}{2} - \frac{\delta_2 - \delta_1}{\sigma^2}\right) \pm \sqrt{\left(\frac{\delta_2 - \delta_1}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\delta_2}{\sigma^2}}. \tag{3.16}$$

Hence, the solution of equation (3.14) must have the following form:

$$f_t(c) = K_1 c^{\beta_+} + K_2 c^{\beta_-}. \tag{3.17}$$

Since the profit flow $\max(c-1, 0)$ of the equation (3.13) is defined differently when $c \geq 1$ and $c < 0$, we solve the equation separately for $c \geq 1$ and $c < 1$, and then, "we stitch together" the two solutions at the point $c = 1$ where the two regions meet.

In the region $c < 1$, we have that the profit flow is zero, and only the homogeneous part of the equation remains. Therefore, the general solution on this region is given by equation (3.17).

In the region $c \geq 1$ the profit flow of the equation (3.13) is present so the general solution is equal to (3.17) plus any particular solution of the full equation. A simple substitution shows that $(\frac{c}{\delta_1} - \frac{1}{\delta_2})$ satisfies the equation.

Therefore, the general solution for the equation (3.13) is of the form:

$$f_t(C_t) = \begin{cases} K_1 c^{\beta_+} + K_2 c^{\beta_-} & , c < 1 \\ K_1^* c^{\beta_+} + K_2^* c^{\beta_-} + \frac{c}{\delta_1} - \frac{1}{\delta_2} & , c \geq 1 \end{cases}. \tag{3.18}$$

The constants in the solutions are determined using considerations that apply at the boundaries of the regions. In the region $c \geq 1$, when c becomes very large, the suspension option is unlikely to be invoked except perhaps in the very remote future, so its value should be zero. For this we should rule out the positive power of c in the solution, by making $K_1^* = 0$. In the region $c < 1$, as c nears zero the expected present value of future profits should then go to zero, and so should the value of the project. For this we should rule out the negative power of c in the solution, by making $K_2 = 0$. Hence the solution is of the form:

$$f_t(C_t) = \begin{cases} K_1 c^{\beta_+} & , c < 1 \\ K_2^* c^{\beta_-} + \frac{c}{\delta_1} - \frac{1}{\delta_2} & , c \geq 1 \end{cases}. \tag{3.19}$$

This still leaves two constants, for which we consider the point $c = 1$ where the two regions meet. Using the value-matching and smooth-pasting conditions, the constants K_2^* and K_1 must solve the following system:

$$\begin{cases} K_2^* \beta_- c^{\beta_- - 1} + \frac{1}{\delta_1} = K_1 \beta_+ c^{\beta_+ - 1} \\ K_2^* c^{\beta_-} + \frac{c}{\delta_1} - \frac{1}{\delta_2} = K_1 c^{\beta_+} \end{cases} \tag{3.20}$$

with $c = 1$

$$\begin{cases} K_2^* \beta_- + \frac{1}{\delta_1} = K_1 \beta_+ \\ K_2^* + \frac{1}{\delta_1} - \frac{1}{\delta_2} = K_1 \end{cases} \tag{3.21}$$

$$\Leftrightarrow \begin{cases} \frac{1}{\beta_- - \beta_+} \left[\frac{\beta_+ - 1}{\delta_1} - \frac{\beta_+}{\delta_2} \right] = K_2^* \\ \frac{1}{\beta_- - \beta_+} \left[\frac{\beta_- - 1}{\delta_1} - \frac{\beta_-}{\delta_2} \right] = K_1 \end{cases} \tag{3.22}$$

Substituting this in (3.19) yields:

$$f_t(c) = \begin{cases} \frac{1}{\beta_- - \beta_+} \left[\frac{\beta_- - 1}{\delta_1} - \frac{\beta_-}{\delta_2} \right] c^{\beta_+} & , c < 1 \\ \frac{1}{\beta_- - \beta_+} \left[\frac{\beta_+ - 1}{\delta_1} - \frac{\beta_+}{\delta_2} \right] c^{\beta_-} + \frac{c}{\delta_1} - \frac{1}{\delta_2} & , c \geq 1 \end{cases} \tag{3.23}$$

We can now unwind the change of variable to get the solution of equation (3.12). Given that $V_t(x, y) = y f_t(c)$, $S_{1,t} = x$, $S_{2,t} = y$ and $\frac{S_{1,t}}{S_{2,t}} = c$

$$V_t(S_{1,t}, S_{2,t}, \infty) = \begin{cases} \frac{1}{\beta_- - \beta_+} \left[\frac{\beta_- - 1}{\delta_1} - \frac{\beta_-}{\delta_2} \right] S_{2,t} \left(\frac{S_{1,t}}{S_{2,t}} \right)^{\beta_+} & , S_{1,t} < S_{2,t} \\ \frac{1}{\beta_- - \beta_+} \left[\frac{\beta_+ - 1}{\delta_1} - \frac{\beta_+}{\delta_2} \right] S_{2,t} \left(\frac{S_{1,t}}{S_{2,t}} \right)^{\beta_-} + \frac{S_{1,t}}{\delta_1} - \frac{S_{2,t}}{\delta_2} & , S_{1,t} \geq S_{2,t} \end{cases} . \tag{3.24}$$

□

Before proceeding to the next section, we will rewrite equation (3.10) under a different notation that will facilitate the derivation of the forward starting perpetuity of the problem at hand:

Definition 3.1. *We define a mapping function $B(x)$ expressed as:*

$$B(x) = \begin{cases} \frac{1}{\beta_- - \beta_+} \left[\frac{\beta_- - 1}{\delta_1} - \frac{\beta_-}{\delta_2} \right] & , x = \beta_+ \\ \frac{1}{\beta_- - \beta_+} \left[\frac{\beta_+ - 1}{\delta_1} - \frac{\beta_+}{\delta_2} \right] & , x = \beta_- \\ \frac{1}{\delta_1} & , x = 1 \\ \frac{1}{\delta_2} & , x = 0 \end{cases} \tag{3.25}$$

Using Definition 3.1 and the indicator function, we can rewrite equation (3.10) as:

$$\begin{aligned} V_t(S_{1,t}, S_{2,t}, \infty) = B(\beta_+) S_{2,t} \left(\frac{S_{1,t}}{S_{2,t}} \right)^{\beta_+} \mathbb{1}_{\{S_{1,t} < S_{2,t}\}} + \\ + \left(B(1) S_{1,t} - B(0) S_{2,t} + B(\beta_-) S_{2,t} \left(\frac{S_{1,t}}{S_{2,t}} \right)^{\beta_-} \right) \mathbb{1}_{\{S_{1,t} \geq S_{2,t}\}}. \end{aligned} \tag{3.26}$$

3.3. Forward start perpetual case

3.3.1. Forward start perpetual real options problem

Since the caplets contained within the integral (3.5) are of European-style, we can compute the finite cap integral from the perpetial cap by subtracting the risk-neutral expectation of the forward start perpetual cap, that is

$$V_t(S_{1,t},S_{2,t},T)=V_t(S_{1,t},S_{2,t},\infty)-e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[V_T(S_{1,T},S_{2,T},\infty)|\mathcal{F}_t\right], \tag{3.27}$$

where the risk-neutral expectation of the forward starting perpetual cap can be expressed as

$$\begin{aligned} e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[V_T(S_{1,T},S_{2,T},\infty)|\mathcal{F}_t\right] &= \int_T^\infty e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[m_T(S_{1,T},S_{2,T},u)|\mathcal{F}_t\right]du \quad (3.28)\\ &= \int_T^\infty m_t(S_{1,t},S_{2,t},u)du. \quad (3.29)\end{aligned}$$

using equation (3.7) and Fubini's theorem.

3.3.2. Forward start perpetual caps

Next proposition produces a formula for valuing the forward expectation of a perpetual claim using the risk-neutral pricing theory.

PROPOSITION 3.2. *The time-t value of a perpetual claim starting at time T is given by:*

$$\begin{aligned} e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[V_T(S_{1,T},S_{2,T},\infty)|\mathcal{F}_t\right] = {} & B(\beta_+)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_+}\Phi\left(-d_{\beta_+}(S_{1,t},S_{2,t},\tau)\right)\\ &+B(1)e^{-\delta_1\tau}S_{1,T}\Phi(d_1(S_{1,t},S_{2,t},\tau))\\ &-B(0)e^{-\delta_2\tau}S_{2,T}\Phi(d_0(S_{1,t},S_{2,t},\tau))\\ &+B(\beta_-)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_-}\Phi\left(d_{\beta_-}(S_{1,t},S_{2,t},\tau)\right), \end{aligned} \tag{3.30}$$

where

$$d_\beta(S_{1,t},S_{2,t},\tau)=\frac{\ln\left(\frac{S_{1,t}}{S_{2,t}}\right)+\left(\delta_2-\delta_1+\sigma_c^2\left(\beta-\frac{1}{2}\right)\right)\tau}{\sigma_c\sqrt{\tau}}. \tag{3.31}$$

PROOF. Let us first write equation (3.26) at the forward time T:

$$\begin{aligned} V_t(S_{1,T},S_{2,T},\infty) = {} & B(\beta_+)S_{2,T}\left(\frac{S_{1,T}}{S_{2,T}}\right)^{\beta_+}\mathbb{1}_{\{S_{1,T}<S_{2,T}\}}+\\ &+\left(B(1)S_{1,T}-B(0)S_{2,T}+B(\beta_-)S_{2,T}\left(\frac{S_{1,T}}{S_{2,T}}\right)^{\beta_-}\right)\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}}. \end{aligned} \tag{3.32}$$

The t-time expected value of this T-time expectation is then, given by:

$$\begin{aligned} e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[V_T(S_{1,T},S_{2,T},\infty)|\mathcal{F}_t\right] = {} & B(\beta_+)e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[S_{2,T}\left(\frac{S_{1,T}}{S_{2,T}}\right)^{\beta_+}\mathbb{1}_{\{S_{1,T}<S_{2,T}\}}|\mathcal{F}_t\right]\\ &+B(1)e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[S_{1,T}\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}}|\mathcal{F}_t\right]\\ &-B(0)e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[S_{2,T}\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}}|\mathcal{F}_t\right]\\ &+B(\beta_-)e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[S_{2,T}\left(\frac{S_{1,T}}{S_{2,T}}\right)^{\beta_-}\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}}|\mathcal{F}_t\right]. \end{aligned} \tag{3.33}$$

We note that $B(\beta_+)$, $B(1)$, $B(0)$ and $B(\beta_-)$ are constants. Using the change of *numéraire* used in Chapter 2:

$$e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[S_{2,T}\left(\frac{S_{1,T}}{S_{2,T}}\right)^{\beta_+}\mathbb{1}_{\{S_{1,T}<S_{2,T}\}}|\mathcal{F}_t\right] = e^{-r\tau}S_{2,t}e^{(r-\delta_2)\tau}\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^{S_2}}{d\mathbb{Q}}C_T^{\beta_+}\mathbb{1}_{\{C_T<1\}}|\mathcal{F}_t\right]$$
$$= S_{2,t}e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^{\beta_+}\mathbb{1}_{\{C_T<1\}}|\mathcal{F}_t\right] \quad (3.34)$$

$$e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[S_{1,T}\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}}|\mathcal{F}_t\right] = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[S_{2,T}C_T\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] =$$
$$= e^{-r\tau}S_{2,t}e^{(r-\delta_2)\tau}\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^{S_2}}{d\mathbb{Q}}C_T\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right]$$
$$= S_{2,t}e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] \quad (3.35)$$

$$e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[S_{2,T}\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}}|\mathcal{F}_t\right] = e^{-r\tau}S_{2,t}e^{(r-\delta_2)\tau}\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^{S_2}}{d\mathbb{Q}}\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right]$$
$$= S_{2,t}e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] \quad (3.36)$$

$$e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[S_{2,T}\left(\frac{S_{1,T}}{S_{2,T}}\right)^{\beta_-}\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}}|\mathcal{F}_t\right] = e^{-r\tau}S_{2,t}e^{(r-\delta_2)\tau}\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^{S_2}}{d\mathbb{Q}}C_T^{\beta_-}\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] =$$
$$= S_{2,t}e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^{\beta_-}\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right]. \quad (3.37)$$

To calculate these expectations we first note that all of these are the expectations of the ratio process C_T^β with $\beta = \beta_+$ in equation (3.34), $\beta = 1$ in equation (3.35), $\beta = 0$ in equation (3.36) and $\beta = \beta_-$ in equation (3.37), thus we can solve the expectations for the process C_T^β with a general elasticity β,

$$\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^\beta\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] \quad (3.38)$$

and

$$\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^\beta\mathbb{1}_{\{C_T<1\}}|\mathcal{F}_t\right] \quad (3.39)$$

and then use them with the appropriated elasticity to solve the expectations (3.34), (3.35), (3.36) and (3.37).

We will solve the expectation (3.38). The solution of the expectation (3.39) is analogous. Equation (2.21) implies that:

$$C_T^\beta = C_t^\beta \exp\left[(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2})\beta\tau - \beta\sigma_c Y\sqrt{\tau}\right], \quad (3.40)$$

where $Y = -\frac{W_{c,T}^{\mathbb{Q}^{S_2}} - W_{c,t}^{\mathbb{Q}^{S_2}}}{\sqrt{\tau}}$ is a standard normal random variable independent of $\mathcal{F}_t$. Substituting this in equation (3.38):

$$\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^\beta\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] = C_t^\beta\mathbb{E}^{\mathbb{Q}^{S_2}}\left[e^{(\delta_2-\delta_1-\frac{\sigma_c^2}{2})\beta\tau-\beta\sigma_c Y\sqrt{\tau}}\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] \quad (3.41)$$

and we have that:

$$C_T \geq 1 \Leftrightarrow d_0(C_t, 1, \tau) = \frac{\ln C_t + \left(\delta_2 - \delta_1 - \frac{\sigma_c^2}{2}\right)\tau}{\sigma_c\sqrt{\tau}} \geq Y. \tag{3.42}$$

With $C_t = x$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^\beta \mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)} x^\beta e^{(\delta_2-\delta_1-\frac{\sigma_c^2}{2})\beta\tau-\beta\sigma_c y\sqrt{\tau}} e^{-\frac{1}{2}y^2}dy \\
&= \frac{x^\beta e^{\left(\delta_2-\delta_1-\frac{\sigma_c^2}{2}\right)\beta\tau}}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)} e^{-\beta\sigma_c y\sqrt{\tau}}e^{-\frac{1}{2}y^2}dy \\
&= \frac{x^\beta e^{\left(\delta_2-\delta_1-\frac{\sigma_c^2}{2}\right)\beta\tau}}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)} e^{-\frac{1}{2}\left(\sigma_c^2\beta^2\tau+2\beta\sigma_c y\sqrt{\tau}+y^2\right)+\frac{\sigma_c^2\beta^2\tau}{2}}dy \\
&= \frac{x^\beta e^{\left(\delta_2-\delta_1-\frac{\sigma_c^2}{2}\right)\beta\tau+\frac{\sigma_c^2\beta^2\tau}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)} e^{-\frac{1}{2}\left(y+\sigma_c\beta\sqrt{\tau}\right)^2}dy.
\end{aligned}$$

Let $z = y + \sigma_c\beta\sqrt{\tau}$, then:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^\beta \mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] &= \frac{x^\beta e^{\left[(\delta_2-\delta_1)\beta+\frac{\sigma_c^2}{2}\beta(\beta-1)\right]\tau}}{\sqrt{2\pi}}\int_{-\infty}^{d_0(x,1,\tau)+\sigma_c\beta\sqrt{\tau}} e^{-\frac{1}{2}(z)^2}dz \\
&= x^\beta e^{\left[(\delta_2-\delta_1)\beta+\frac{\sigma_c^2}{2}\beta(\beta-1)\right]\tau}\Phi(d_0(x,1,\tau)+\sigma_c\beta\sqrt{\tau}).
\end{aligned} \tag{3.43}$$

Now, we can include the discount factor $e^{-\delta_2\tau}$ in equation (3.43) which yields:

$$\begin{aligned}
e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^\beta \mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] &= e^{-\delta_2\tau}x^\beta e^{\left[(\delta_2-\delta_1)\beta+\frac{\sigma_c^2}{2}\beta(\beta-1)\right]\tau}\Phi(d_0(x,1,\tau)+\sigma_c\beta\sqrt{\tau}) \\
&= x^\beta e^{\left[(\delta_2-\delta_1)\beta+\frac{\sigma_c^2}{2}\beta(\beta-1)-\delta_2\right]\tau}\Phi(d_0(x,1,\tau)+\sigma_c\beta\sqrt{\tau}).
\end{aligned} \tag{3.44}$$

Since $x = C_t$,

$$e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^\beta\, \mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] = e^{q(\beta)\tau}C_t^\beta\Phi\left(d_\beta(C_t,1,\tau)\right), \tag{3.45}$$

where

$$d_\beta(C_t,1,\tau) = \frac{\ln\frac{C_t}{1} + \left(\delta_2-\delta_1+\sigma_c^2\left(\beta-\frac{1}{2}\right)\right)\tau}{\sigma_c\sqrt{\tau}} \tag{3.46}$$

and

$$q(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + (\delta_2-\delta_1)\beta - \delta_2. \tag{3.47}$$

It can be shown analogously

$$e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^\beta\, \mathbb{1}_{\{C_T< 1\}}|\mathcal{F}_t\right] = e^{q(\beta)\tau}C_t^\beta\Phi\left(-d_\beta(C_t,1,\tau)\right). \tag{3.48}$$

We can now use the general equations (3.45) and (3.48) to solve equations (3.34), (3.35), (3.36) and (3.37). Using equation (3.48) with $\beta = \beta_+$ for equation (3.34):

$$S_{2,t}e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^{\beta_+}\mathbb{1}_{\{C_T<1\}}|\mathcal{F}_t\right] = S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_+}\Phi\left(-d_{\beta_+}(S_{1,t},S_{2,t},\tau)\right) \tag{3.49}$$

using equation (3.45) with $\beta = 1$ for equation (3.35):

$$S_{2,t}e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] = e^{-\delta_1\tau}S_{1,t}\Phi\left(d_1(S_{1,t},S_{2,t},\tau)\right) \tag{3.50}$$

using equation (3.45) with $\beta = 0$ for equation (3.36):

$$S_{2,t}e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] = e^{-\delta_2\tau}S_{2,t}\Phi\left(d_0(S_{1,t},S_{2,t},\tau)\right) \tag{3.51}$$

using equation (3.45) with $\beta = \beta_-$ for equation (3.37):

$$S_{2,t}e^{-\delta_2\tau}\mathbb{E}^{\mathbb{Q}^{S_2}}\left[C_T^{\beta_-}\mathbb{1}_{\{C_T\geq 1\}}|\mathcal{F}_t\right] = S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_-}\Phi\left(d_{\beta_-}(S_{1,t},S_{2,t},\tau)\right). \tag{3.52}$$

Finally combining equations (3.34), (3.35), (3.36), (3.37), (3.49), (3.50), (3.51) and (3.52), equation (3.33) becomes:

$$\begin{aligned} e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[V_T(S_{1,T},S_{2,T},\infty)|\mathcal{F}_t\right] = {} & B(\beta_+)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_+}\Phi\left(-d_{\beta_+}(S_{1,t},S_{2,t},\tau)\right) \\ & + B(1)e^{-\delta_1\tau}S_{1,T}\Phi(d_1(S_{1,t},S_{2,t},\tau)) \\ & - B(0)e^{-\delta_2\tau}S_{2,T}\Phi(d_0(S_{1,t},S_{2,t},\tau)) \\ & + B(\beta_-)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_-}\Phi\left(d_{\beta_-}(S_{1,t},S_{2,t},\tau)\right). \end{aligned} \tag{3.53}$$

□

3.4. Finite caps

Finally we will combine the results from Section 3.2.2 and Section 3.3.2 to obtain the formula for the finite case.

Proposition 3.3. *The t-value of the finite-lived claim is:*

$$\begin{aligned} V_t(S_{1,t},S_{2,t},T) = {} & B(\beta_-)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_-}[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - \Phi(d_{\beta_-}(S_{1,t},S_{2,t},\tau))] \\ & + B(1)S_{1,t}\left[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - e^{-\delta_1\tau}\Phi(d_1(S_{1,t},S_{2,t},\tau))\right] \\ & - B(0)S_{2,t}\left[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - e^{-\delta_2\tau}\Phi(d_0(S_{1,t},S_{2,t},\tau))\right] \\ & - B(\beta_+)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_+}[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - \Phi(d_{\beta_+}(S_{1,t},S_{2,t},\tau))]. \end{aligned} \tag{3.54}$$

Proof. Substituting equations (3.26) and equation (3.30) into equation (3.6):

$$\begin{aligned} V_t(S_{1,t}, S_{2,t}, T) &= V_t(S_{1,t}, S_{2,t}, \infty) - e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[V_T(S_{1,T}, S_{2,T}, \infty)\right] \\ &= B(\beta_-)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_-}\left[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - \Phi(d_{\beta_-}(S_{1,t}, S_{2,t}, \tau))\right] \\ &+ B(1)S_{1,t}\left[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - e^{-\delta_1\tau}\Phi(d_1(S_{1,t}, S_{2,t}, \tau))\right] \\ &- B(0)S_{2,t}\left[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - e^{-\delta_2\tau}\Phi(d_0(S_{1,t}, S_{2,t}, \tau))\right] \\ &+ B(\beta_+)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_+}\left[\mathbb{1}_{\{S_{1,T}< S_{2,T}\}} - \Phi(-d_{\beta_+}(S_{1,t}, S_{2,t}, \tau))\right]. \end{aligned} \tag{3.55}$$

Using the relationship

$$\mathbb{1}_{\{S_{1,T}<S_{2,T}\}} - \Phi(-d_\beta(S_{1,t}, S_{2,t}, \tau)) = -(\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - \Phi(d_\beta(S_{1,t}, S_{2,t}, \tau))) \tag{3.56}$$

we get

$$\begin{aligned} V_t(S_{1,t}, S_{2,t}, T) &= B(\beta_-)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_-}\left[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - \Phi(d_{\beta_-}(S_{1,t}, S_{2,t}, \tau))\right] \\ &+ B(1)S_{1,t}\left[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - e^{-\delta_1\tau}\Phi(d_1(S_{1,t}, S_{2,t}, \tau))\right] \\ &- B(0)S_{2,t}\left[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - e^{-\delta_2\tau}\Phi(d_0(S_{1,t}, S_{2,t}, \tau))\right] \\ &- B(\beta_+)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_+}\left[\mathbb{1}_{\{S_{1,T}\geq S_{2,T}\}} - \Phi(d_{\beta_+}(S_{1,t}, S_{2,t}, \tau))\right]. \end{aligned} \tag{3.57}$$

□

3.5. Finite floors

The difference between the cap and the floor is the swap (a result that depends on caplet and floorlet parity). Thus floors $F_t(S_{1,t}, S_{2,t}, T)$ are valued as a function of the corresponding cap less a swap of the revenues associated with $S_{1,t}$ against those with $S_{2,t}$, both over a finite horizon T. Using equation (3.54) and the relationship (3.56)

$$\begin{aligned} F_t(S_{1,t}, S_{2,t}, T) &= V_t(S_{1,t}, S_{2,t}, T) - B(1)S_{1,t}(1 - e^{-\delta_1\tau}) + B(2)S_{2,t}(1 - e^{-\delta_2\tau}) \\ &= -B(\beta_-)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_-}\left[\mathbb{1}_{\{S_{1,T}< S_{2,T}\}} - \Phi(-d_{\beta_-}(S_{1,t}, S_{2,t}, \tau))\right] \\ &- B(1)S_{1,t}\left[\mathbb{1}_{\{S_{1,T}< S_{2,T}\}} - e^{-\delta_1\tau}\Phi(-d_1(S_{1,t}, S_{2,t}, \tau))\right] \\ &+ B(0)S_{2,t}\left[\mathbb{1}_{\{S_{1,T}< S_{2,T}\}} - e^{-\delta_2\tau}\Phi(-d_0(S_{1,t}, S_{2,t}, \tau))\right] \\ &+ B(\beta_+)S_{2,t}\left(\frac{S_{1,t}}{S_{2,t}}\right)^{\beta_+}\left[\mathbb{1}_{\{S_{1,T}< S_{2,T}\}} - \Phi(-d_{\beta_+}(S_{1,t}, S_{2,t}, \tau))\right]. \end{aligned} \tag{3.58}$$

It is also interesting to note that, the value of the floor is the same the cap on $S_{2,t}$ with strike $S_{1,t}$:

$$F_t(S_{1,t}, S_{2,t}, T) = V_t(S_{2,t}, S_{1,t}, T). \tag{3.59}$$

In this case, the *numéraire* to price $V_t(S_{2,t}, S_{1,t}, T)$, will be $S_{1,t}$ instead of $S_{2,t}$.

CHAPTER 4

Conclusions

We began this thesis by exploring the key concepts of an option to exchange one asset for another where the assets pay dividends, the orthogonalization of the pricing problem and the change of *numéraire* to reduce the problem to the one dimensional case. We then tackled the problem of the continuous sum of such exchange options using the time decomposition technique presented by Shackleton & Wojakowski (2007) to reach closed-form solutions.

The valuation of finite maturity caps, floors, and collars on continuous flows has typically been addressed using methods inspired by real options literature, which often focus on perpetual solutions. In our approach, the problem is initially solved for the perpetual case, and then the finite maturity case is derived. This is done by subtracting the risk-neutral expectation of the forward start perpetual solution from the corresponding current perpetual solution. Dias, Nunes & Silva (2024) have solved this problem solving the integral (3.5) using the process of integration by parts.

As prospects of future research, one could apply the framework developed in this thesis to price finite horizon situations where switching can occur frequently and repeatedly between two randomly variable flows: One use case of this framework would be a project to generate energy, where the price of energy and the fuel to generate it are traded assets. In this framework one could stop production of energy if the price of the fuel would be higher then the profit of selling that energy. This could be done instantaneously, at any moment, and with no cost. One could use the close form solutions of this thesis to derive the so-called Greeks to find hedging strategies for these projects. Another topic of future research would be to value the option to invest in such a project.

References

Black, F. 1976. “The Pricing of Commodity Contracts.” Journal of Financial Economics 3: 167–179.

Black, F., & M. Scholes. 1973. “The Pricing of Options and Corporate Liabilities.” Journal of Political Economy 81, no. 3: 637–654.

Dias, J.C., Nunes, J.P.V. & Silva, F.C.d. (2024), ”Novel Analytic Representations for Caps, Floors, Collars, and Exchange Options on Continuous Flows, Arbitrage-Free Relations, and Optimal Investments.” Journal of Futures Markets

Dixit, A. K., & Pindyck, R. S. 1994. ”Investment under Uncertainty”. Princeton University Press.

Margrabe, W. 1978. “The Value of an Option to Exchange One Asset for Another.” Journal of Finance 33, no. 1: 177–186.

McDonald, R. L., and D. R. Siegel. 1985. “Investment and the Valuation of Firms When There Is an Option to Shut Down.” International Economic Review 26, no. 2: 331–349.

Merton, R. C. 1973. “Theory of Rational Option Pricing.” Bell Journal of Economics and Management Science 4, no. 1: 141–183.

Shackleton, M. B., & R. Wojakowski. 2007. “Finite Maturity Caps and Floors on Continuous Flows.” Journal of Economic Dynamics and Control 31: 3843–3859.

Shreve, S., 2004, ”Stochastic Calculus for Finance II: Continuous-Time Models” (Springer, New York).