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Optimal Control Synchronization of a Complex Network of Predator-Prey Systems

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Summary. In this work, we consider a complex network of predator-prey systems, modeling the ecological dynamics of interacting species living in a fragmented environment. We consider non-identical instances of a Lotka-Volterra model with Holling type II functional response. We study optimal control problems, for the minimization of the default of synchronization in the complex network, where the controls reproduce the implementation of ecological corridors. The main goal is to restore biodiversity of life species in a heterogeneous habitat by reaching at least a global coexistence equilibrium, or in a better scenario, a global limit cycle which would guarantee biological oscillations, which means rich life dynamics.

13.1 Introduction

Optimizing the biodiversity restoration of life species in a fragmented habitat through the implementation of ecological corridors between each component of the fragmented environment, while maintaining human activity at a reasonable level, is a challenge that we wish to study in this work. We assume that the geographical habitat of the species is perturbed by the anthropic extension, so that it is fragmented in several patches. This fragmentation is likely to alter the equilibrium of the ecological system. In order to model such a fragmented environment, we consider a complex network of predator-prey models, first proposed in [16], which reproduces the heterogeneous natural environment, that is perturbed by fragmentation, by coupling several patches on which interacting wild species are living. To construct the complex network model, on each patch, the ecological inter-species dynamics are modeled by a Lotka-Volterra predator-prey model with Holling type II functional response, which is able to describe several biological dynamics, such as extinction, coexistence or ecological cycles (see e.g. [21, 31, 33]). Here, each patch can admit its own dynamic, that is, the local components of the network can for instance exhibit an extinction equilibrium on some places, whereas other places can present cycles [16]. Moreover, migrations of biological individuals in space, between each component of the fragmented environment, are taken into account by coupling the patches of the network (see Figure 13.1, where the disks model the patches of a fragmented environment, where the inter-species dynamics of Lotka-Volterra type occur, and the arrows model the ecological corridors which can be implemented between these patches, so as to increase the migrations in space of the species between each patch).

In [16], sufficient conditions of synchronization of the local dynamics, under a variation of the couplings, are proved, namely a theorem for near-synchronization is proved, which guarantees that the complex network remains in a neighborhood of a synchronization state, provided the coupling strength is strong enough, even if the local behaviors are non-identical. This result improves the sufficient conditions of synchronization for the particular case of identical dynamics, proved in [15]. The relevance of synchronization in complex networks has been highlighted by several studies in different areas, such that, coupled oscillators, networks of chemical reactions, neural networks or meta-populations models (see for instance [3, 5, 8, 29] and the references therein).

The main goal of this paper is to optimize the synchronization of the complex network, through optimal control theory. The possibility to reach synchronization through an optimal control process has been studied in [14], with an application to an epidemic model, or in [13], with an application to a panic model. Meanwhile, the dynamics of Lotka-Volterra type models have been widely analyzed (see for instance [7] or [24]) and the optimal control of such models has been studied in [18, 22], but not in the framework of complex networks. On the optimal control of periodic solutions, the non-existence of limit cycle was proved in [9], and periodic optimal control problems have been analyzed in [6, 20, 27, 38].

Focusing on Lotka Volterra models, in [32], a fish population optimal control problem is studied considering the Lotka Volterra model

$$\begin{cases} \dot{x}_1(t) = x_1(t) - x_1(t)x_2(t) \\ \dot{x}_2(t) = -x_2(t) + x_1(t)x_2(t) \\ x_i(t_0) = x_{i0}\,, \quad i = 1, 2\,, \end{cases} \tag{13.1}$$

where $x_1(t)$ and $x_2(t)$ represent the biomass of the prey and predator species, respectively, with initial state conditions $x_i(t_0) = x_{i0}$, $i = 1, 2$.

The main goal in [32] is to bring the control system (13.1) close to a steady state to avoid the high fluctuations that cause economical problems. More precisely, the authors choose to vary the fishing quota for a certain time span $T - t_0$. Adding an objective functional that punishes deviation from the steady state $\tilde{x} = (1, 1)^T$ for $u(t) = 0$, and $\tilde{x} = (1 + c_2, 1 - c_1)^T$ for $u(t) = 1$, respectively. The following optimal control problem is analyzed

$$\min_u \int_{t_0}^{T} (x_1(t) - 1)^2 + (x_2(t) - 1)^2 \, dt$$

such that

$$\begin{cases} \dot{x}_1(t) = x_1(t) - x_1(t)x_2(t) - c_1 x_1(t)u(t) \\ \dot{x}_2(t) = -x_2(t) + x_1(t)x_2(t) - c_2 x_2(t)u(t) \\ x_i(t_0) = x_{i0}\,, \quad u(t) \in [0, 1]\,. \end{cases} \tag{13.2}$$

and where control function $u(t)$ describes the percentage of the fleet that is actually fished at time t. The parameters c_1 and c_2 indicate how many fish would be caught by the entire fleet. For optimization methods to solve the previous optimal control problem see, e.g. [32] and also [40].

In [23], the turnpike phenomenon is illustrated by studying an optimal control problem, considering the control system (13.2) with $c_1 = 0.4$ and $c_2 = 0.2$, initial conditions $(x_1(0), x_2(0)) = (0.5; 0.7)$, final time $T = 60$, and with the following cost functional:

$$\min_u \frac{1}{2} \int_0^T (x_1(t) - 1)^2 + x_2(t)^2 + u(t)^2 \, dt \, .$$

An analogous control system is also considered in [17], where optimal strategies for reaching fixed steady states, namely co-existence of both species are studied. General turnpike results for optimal control problems have been established in [35, 36, 37].

In [4], a Mayer-type optimal control problem is studied for Lotka-Volterra systems with a hunter population, where the goal is to maximize the population of both species at the final time, that is, $x_1(T) + x_2(T)$ and the control u represents the hunting proportionality factor. In [39] the authors analyze analytically a Mayer-type optimal control problem applied to a two dimensional Lotka Volterra system. In [10], the shooting method is applied to a minimal time optimal control problem with the control system from [23, 32].

Recently, advances on geometrical optimal control theory of Generalized Lotka-Volterra systems applied to the intestinal microbiome have been developed in [11, 12].

In this paper, we aim to study a more general problem than the ones studied in [23, 32] reaching at least a global coexistence equilibrium, or in a better scenario, a global limit cycle, instead of a fixed steady state. The optimal control of limit cycles in medical models applied to diabetes and heart attack problem was studied in [19] and [27], respectively. Moreover, the nonexistence of limit cycle for an optimal control problem applied to a diabetes model was proved in [9]. In this chapter, we first consider a controlled complex network of Lotka-Volterra systems, where the strength of the migrations of biological individuals in system (13.6) is replaced by control functions, reproducing the implementation of ecological corridors We prove that a solution of the controlled complex network can reach a near-synchronization state, under sufficient conditions which highlight the importance to consider a positive lower on the controls functions. After, we study optimal control problems where the main goal is the minimization of the default of synchronization in the complex network. We consider different cost functionals taking into account that the dynamics of the controlled complex network ensure the conservation of both species, namely, our goal is to impose synchronization or synchronization of limit cycles. Therefore, the solutions of the optimal control problems lead to a restoration of the biodiversity of life species in a heterogeneous habitat by reaching at least a global coexistence equilibrium, or in a better scenario, a global limit cycle which would guarantee biological oscillations, which means rich life dynamics.

This chapter is organized as follows. In Section 13.2, we recall the construction of the uncontrolled complex network of Lotka-Volterra systems and the near synchronization results, from [16]. In Section 13.3, we propose a controlled complex network, where the strength of the migrations of biological individuals in the Lotka-Volterra systems is replaced by control functions, and prove a sufficient condition for the near-synchronization of the solutions of the controlled system. In Section 13.4, we consider optimal control problems, in order to exert a command on the global behavior of the controlled complex network. To model the goal of restoring biodiversity and biological dynamics in a fragmented environment, we define appropriate cost functionals where the conservation of species is guaranteed by imposing synchronization or synchronization of limit cycles. We end this chapter with Section 13.5 with some conclusions and future work.

13.2 Setting of the Complex Network of Lotka-Volterra Systems

Based on the previous work [16], we present the construction of a complex network of Lotka-Volterra systems, which describes the dynamics of interacting species living in a fragmented environment, and recall important near synchronization results, proved in [16], for the uncontrolled complex network.

13.2.1 Lotka-Volterra predator-prey model with Holling type II functional response

Let us consider a biological environment in which two species interact. We assume that the densities of the species are determined by a predator-prey model of Lotka-Volterra type, which can be written by:

$$\begin{cases} \dot{x} = rx(1-x) - \dfrac{cxy}{\alpha + x}, \\ \dot{y} = -dy + \dfrac{cxy}{\alpha + x}. \end{cases} \tag{13.3}$$

Here, x and y denote the prey and predator density, respectively; $\dot{x}$ and $\dot{y}$ denote their derivatives with respect to the time variable t. The parameters r, c, d and α are positive coefficients; r is the birth rate of the preys, d is the mortality rate of predators, and c, α determine the non-linear interaction between preys and predators (see, for instance [24], for a deep study on the dynamics of predator-prey system (13.3)). Depending on the values of the parameters r, c, d, α, the solutions of system (13.3) can be attracted to a coexistence equilibrium, to an extinction equilibrium or to a limit cycle. The extinction equilibrium is denoted $E_0 = (0, 0)$. The coexistence equilibrium E_1, which implies persistence of each specie, is given, for $c \neq d$, by

$$E_1 = \left(\frac{\alpha d}{c-d}, \frac{r\alpha}{c-d} \left(1 - \frac{\alpha d}{c-d} \right) \right). \tag{13.4}$$

System (13.3) also admits the equilibrium $E_2 = (1, 0)$. Let us introduce the critical value α_0 given by

$$\alpha_0 = \frac{c-d}{c+d}. \tag{13.5}$$

It is well-known (see for instance [24], Chapter 3 or [7], Section 3.4.1) that system (13.3) undergoes a Hopf bifurcation at $\alpha = \alpha_0$. For $\alpha < \alpha_0$, a stable limit cycle bifurcates from the persistence equilibrium E_1. Therefore, for α small enough, system (13.3) presents oscillations, which are interpreted as healthy ecological cycles.

13.2.2 Complex network of predator-prey models for a fragmented environment

Next, we assume that the geographical habitat of the species is perturbed by the anthropic extension, so that it is fragmented in several patches. This fragmentation is likely to alter the equilibrium of the ecological system. In order to model such a fragmented environment, we construct a complex network of predator-prey models as follows.

First, let $n > 0$ denote the number of patches on the fragmented environment. On each patch $i \in \{1, \dots, n\}$, we denote by (x_i, y_i) the densities of preys and predators respectively. We assume that each patch $i \in \{1, \dots, n\}$ can be connected to other patches and we denote by $\mathscr{N}_i \subset \{1, \dots, n\}$ the set of patches which are connected to patch i. We assume that migrations of biological individuals can occur between two connected patches, at rates σ_1 for preys and σ_2 for predators. In this way, the dynamics of the fragmented environment are determined by the following complex network:

$$\begin{cases} \dot{x}_i = r_i x_i(1-x_i) - \dfrac{c_i x_i y_i}{\alpha_i + x_i} - \sigma_1 \displaystyle\sum_{j\in\mathscr{N}_i}(x_i - x_j), \\ \dot{y}_i = -d_i y_i + \dfrac{c_i x_i y_i}{\alpha_i + x_i} - \sigma_2 \displaystyle\sum_{j\in\mathscr{N}_i}(y_i - y_j), \end{cases} \tag{13.6}$$

for $1 \leq i \leq n$, with $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$.

We emphasize that the parameters r_i, c_i, d_i, α_i can differ from one patch to another, which means that the ecological dynamics are non-identical within the fragmented environment. For instance, some patches could present limit cycles, whereas other patches could exhibit an extinction of both species. Note also that the couplings are symmetric, which means that if the species x_i, y_i of patch i can move towards some patch j, then the species x_j, y_j of patch j can conversely move towards patch i.

One remarkable case of fragmented environment is that of a complete graph topology, for which we have $\mathscr{N}_i = \{1,\ldots,n\}\setminus\{i\}$; this situation means that each patch is connected to all other patches. At the opposite, if the coupling parameters σ_1, σ_2 are equal to 0, then no migration of individuals occur in the network.

Let us now introduce some notations. Let $X = \big((x_1,y_1),\ldots,(x_n,y_n)\big)^\top \in \mathbb{R}^{2n}$. For each $i \in \{1,\ldots,n\}$, we denote

$$\begin{aligned} \lambda_i &= (r_i, c_i, d_i, \alpha_i)^\top \in \mathbb{R}^4, \\ f_1(x_i, y_i, \lambda_i) &= r_i x_i(1-x_i) - \frac{c_i x_i y_i}{\alpha_i + x_i}, \\ f_2(x_i, y_i, \lambda_i) &= -d_i y_i + \frac{c_i x_i y_i}{\alpha_i + x_i}, \\ g_1(x_i, X, \sigma_1) &= -\sigma_1 \sum_{j\in\mathscr{N}_i}(x_i - x_j), \\ g_2(y_i, X, \sigma_2) &= -\sigma_2 \sum_{j\in\mathscr{N}_i}(y_i - y_j). \end{aligned} \tag{13.7}$$

We also denote $\sigma = (\sigma_1, \sigma_2)^\top \in \mathbb{R}^2$ and

$$\begin{aligned} &\Lambda = (\lambda_1,\ldots,\lambda_n)^\top \in \mathbb{R}^{4n}, \\ &F(X,\Lambda) = \Big(f_1(x_1,y_1,\lambda_1), f_2(x_1,y_1,\lambda_1),\ldots,f_1(x_n,y_n,\lambda_n), f_2(x_n,y_n,\lambda_n)\Big)^\top \in \mathbb{R}^{2n}, \\ &G(X,\sigma) = \Big(g_1(x_1,X,\sigma_1), g_2(y_1,X,\sigma_2),\ldots,g_1(x_n,X,\sigma_1), g_2(y_n,X,\sigma_2)\Big)^\top \in \mathbb{R}^{2n}. \end{aligned} \tag{13.8}$$

With these notations, the complex network (13.6) can be written under the following short form

$$\dot{X} = F(X,\Lambda) + G(X,\sigma). \tag{13.9}$$

13.2.3 Review of known results

In this section, we recall recent results obtained in [16], that motivate the controlled system and the optimal control problem studied in the present work.

The following theorem guarantees that the complex network problem determined by system (13.9) admits global solutions.

Theorem 1 ([16]). *Let $X_0 \in (\mathbb{R}^+)^{2n}$. Then the complex network problem determined by* (13.9) *and $X(0) = X_0$ admits a unique global solution $X(t, X_0)$ defined on $[0, +\infty)$, whose components are non-negative.*

Furthermore, the flow induced by Equation (13.9) *admits a positively invariant region Θ which is compact in $(\mathbb{R}^+)^{2n}$.*

One remarkable property of complex network is the *synchronization* property. The following definition is classical.

Definition 1 (Synchronization). *Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. We say that the patches i and j of the complex network* (13.9) *synchronize in Θ if, for any initial condition $X_0 \in \Theta$, the solution of* (13.9) *starting from X_0 satisfies*

$$\lim_{t \to +\infty} \Big(|x_i(t) - x_j(t)|^2 + |y_i(t) - y_j(t)|^2 \Big) = 0.$$

We say that the complex network (13.9) *synchronizes in Θ if every pair (i, j) of patches synchronizes in Θ.*

In the case of a complex network of nonidentical systems, it is not always possible to prove that a synchronization state is reached. Therefore, we are led to introduce a relaxed definition of synchronization, called *near-synchronisation*.

Definition 2 (Near-synchronization). *Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. We say that the patches i and j of the complex network* (13.9) *nearly synchronize in Θ with respect to $\tilde{\sigma}$ if, for any initial condition $X_0 \in \Theta$, and for any $\varepsilon > 0$, the solution of* (13.9) *starting from X_0 satisfies*

$$0 \leq \lim_{t \to +\infty} \Big(|x_i(t) - x_j(t)|^2 + |y_i(t) - y_j(t)|^2 \Big) < \varepsilon,$$

for $\tilde{\sigma}$ sufficiently large.

We say that the complex network (13.9) *nearly synchronizes in Θ if every pair (i, j) of patches nearly synchronizes in Θ.*

In [16], sufficient conditions of near-synchronization have been established for the complex network (13.9) with non-identical systems. We recall below these results. For the sake of simplicity, it is assumed that the graph underlying the complex network (13.9) is complete, that is, each patch is connected to all other patches; equivalently, we have $\mathscr{N}_i = \{1, \ldots, n\} \setminus \{i\}$ for $1 \leq i \leq n$, where $\mathscr{N}_i$ denotes the finite set of patches which are connected to patch i. For all $i, j \in \{1, \ldots, n\}$, we introduce the energy function $E_{i,j}$ defined along the trajectories of the complex network by

$$E_{i,j}(t) = \frac{1}{2}\Big[|x_i(t) - x_j(t)|^2 + |y_i(t) - y_j(t)|^2 \Big], \tag{13.10}$$

and for $\lambda_i = (r_i, d_i, c_i, \alpha_i), \lambda_j = (r_j, d_j, c_j, \alpha_j) \in \mathbb{R}^4$, denote

$$\|\lambda_i - \lambda_j\|_\infty = \max \Big\{ |r_i - r_j|, |d_i - d_j|, |c_i - c_j|, |\alpha_i - \alpha_j| \Big\}.$$

The next theorem establishes an estimate of the energy functions $E_{i,j}$ defined by (13.10).

Theorem 2 (Near-synchronization of the uncontrolled complex network predator-prey model, [16]). *There exist positive constants* η, δ *such that, for any initial condition* $X_0 \in \Theta$, *the solution of the complex network* (13.9) *starting from* X_0 *satisfies*

$$\dot{E}_{i,j}(t) \leq \eta \left\|\lambda_i - \lambda_j\right|_\infty E_{i,j}^{1/2}(t) + \Big[\delta - 2n\tilde{\sigma}\Big] E_{i,j}(t), \quad t > 0, \tag{13.11}$$

where $\tilde{\sigma} = \min\{\sigma_1, \sigma_2\}$.[3]

Furthermore, the constants η *and* δ *do not depend on the coupling parameters* σ_1, σ_2.

We now recall important consequences of Theorem 2.

Corollary 1 ([16]). *Assume that* $\lambda_i = \lambda_j$ *for some* $i, j \in \{1, \ldots, n\}$. *Then the patches* i *and* j *synchronize if the following condition is fulfilled:*

$$2n\tilde{\sigma} > \delta. \tag{13.12}$$

If $\lambda_i = \lambda_j$ for all $i, j \in \{1, \ldots, n\}$, then obviously the whole network synchronizes under condition (13.12). Next, since the constant δ does not depend on the coupling parameters σ_1, σ_2, the sufficient condition (13.12) can easily be satisfied, provided the number n of patches in the network is sufficiently large, or provided the minimum coupling strength $\tilde{\sigma} = \min\{\sigma_1, \sigma_2\}$ is sufficiently large.

From the ecological point of view, increasing the number n of patches in the network would correspond to a worse fragmentation of the habitat, which is not a reasonable strategy for our purposes. However, increasing the minimum coupling strength $\tilde{\sigma}$ can be realized by providing wider ecological corridors.

The non trivial case of Theorem 2 corresponds to a complex network of non-identical patches, for which we have $\lambda_i \neq \lambda_j$ for at least one pair $(i, j) \in \{1, \ldots, n\}^2$. In that case, the synchronization state $\{(x_i, y_i) = (x_j, y_j)\}$ is likely to present a soft loss of stability. Indeed, it is well-known that the solution w of the Bernoulli equation

$$\dot{w}(t) = \eta \left\|\lambda_i - \lambda_j\right|_\infty w^{1/2}(t) + \Big[\delta - 2n\tilde{\sigma}\Big] w(t), \quad t > 0, \tag{13.13}$$

converges towards a positive limit given by

$$\lim_{t \to +\infty} w(t) = \left(\frac{\eta \left\|\lambda_i - \lambda_j\right|_\infty}{\delta - 2n\tilde{\sigma}}\right)^2,$$

provided $w(0) > 0$. We obtain the following corollaries.

Corollary 2 ([16]). *The energy function* $E_{i,j}$ *defined by* (13.10) *along a solution of the complex network* (13.9) *starting from* $X_0 \in \Theta$, *satisfies*

$$0 \leq \limsup E_{i,j}(t) \leq \left(\frac{\eta \left\|\lambda_i - \lambda_j\right|_\infty}{\delta - 2n\tilde{\sigma}}\right)^2. \tag{13.14}$$

complex

[3] In this paper, we correct a misprint of [16], since the quantity $2(n-1)\tilde{\sigma}$ in [16] should be $2n\tilde{\sigma}$.

Corollary 3 ([16]). *The complex network* (13.9) *nearly synchronizes in Θ with respect to the the minimum coupling strength $\tilde{\sigma}$.*

Remark 1. Note that near-synchronization can occur in the complex network (13.9) without imposing any particular asymptotic dynamics; for example, the complex network could synchronize towards a global dynamic of extinction, towards a global dynamic of coexistence, or towards a global dynamic of limit cycles (it could even happen that a new dynamic emerges from the complex network structure).

In the next section, we construct a controlled complex network of Lotka-Volterra systems, where the strength of the migrations of biological individuals in system (13.6) is replaced by control functions. We prove that a solution of the controlled system can reach a near-synchronization state, under sufficient conditions which highlight the importance to consider a positive lower bound on the controls functions.

13.3 Controlled Synchronization

In this section, we present a controlled complex network of Lotka-Volterra systems, where the strength of the migrations of biological individuals in system (13.6) is replaced by control functions $u_{i,j}(\cdot) \in L^\infty(0,T)$, $1 \leq i,j \leq n$.

The main goal is to restore biodiversity and biological dynamics in a fragmented environment. Our aim is to reach at least a global coexistence equilibrium, or better, a global limit cycle which would guaranty biological oscillations, which means rich life dynamics.

13.3.1 Setting of the control system

We consider the control system, given by

$$\begin{cases} \dot{x}_i = r_i x_i(1 - x_i) - \dfrac{c_i x_i y_i}{\alpha_i + x_i} - \displaystyle\sum_{j \in \mathcal{N}_i} u_{i,j}(t)(x_i - x_j), \\ \dot{y}_i = -d_i y_i + \dfrac{c_i x_i y_i}{\alpha_i + x_i} - \displaystyle\sum_{j \in \mathcal{N}_i} u_{i,j}(t)(y_i - y_j), \end{cases} \tag{13.15}$$

for $1 \leq i \leq n$, with the following control constraints

$$u_{\min} \leq u_{i,j}(t) \leq u_{\max} \quad \forall t \in [0,T], \quad \text{for} \quad \text{all} \quad (i,j) \in \{1,\ldots,n\}^2, \tag{13.16}$$

with $u_{\min} > 0$. Hence, the set of admissible control functions is given by

$$\Omega = \left\{ u_{i,j}(\cdot) \in L^\infty(0,T) \mid u_{\min} \leq u_{i,j}(t) \leq u_{\max} \quad \forall t \in [0,T], \forall \quad (i,j) \in \{1,\ldots,n\}^2 \right\}.$$

Moreover, we consider fixed initial conditions $X(0) = X_0 \in (\mathbb{R}^+)^{2n}$.

Analogously to equation (13.9), we can write the controlled system (13.15) in the form

$$\dot{X} = F(X, \Lambda) + G(X, \{u_{i,j}\}_{1 \leq i,j \leq n}).$$

The following theorem guarantees the existence of a positively invariant region for a solution of the controlled system (13.15).

Let $a_0 = \sum_{i=1}^{n} a_i$, $b_0 = \min_{1\leq i\leq n} b_i$, $d_0 = \min_{1\leq i\leq n} d_i$,$c_0 = \min\{b_0, d_0\}$, where the coefficients a_i, b_i are chosen such that

$$r_i s(1-s) \leq a_i - b_i s,$$

for all $s \in \mathbb{R}$.

Theorem 3 (Positively invariant region). *The region Θ defined by*

$$\Theta = \left\{ X = (x_i, y_i)_{1\leq i\leq n} \in (\mathbb{R}^+)^{2n} \;\middle|\; \sum_{1\leq i\leq n} (x_i + y_i) \leq \frac{a_0}{c_0} \right\} \tag{13.17}$$

is positively invariant for the flow induced by the controlled system (13.15).

Proof. Let $P(t) = \sum_{1\leq i\leq n} \big(x_i(t) + y_i(t)\big)$. Summing the equations of the complex network problem, we easily prove that

$$\dot{P} + c_0 P \leq a_0,$$

since the sum of the control couplings vanishes. Applying Gronwall lemma finishes the proof.

13.3.2 Near-synchronization of the controlled system

In this section, we prove that a solution of the controlled system (13.15) can reach a near-synchronization state, under sufficient conditions which highlight the importance to consider a positive lower bound on the controls functions. The following theorem establishes an estimate of the energy function corresponding to a solution of the control system (13.15).

Theorem 4 (Energy estimate the controlled system). *Assume that the graph $\mathscr{G}$ underlying the complex network* (13.15) *is a complete graph. Then the energy functions $E_{i,j}$ defined by*

$$E_{i,j}(t) = \frac{1}{2}\big[(x_i - x_j)^2 + (y_i - y_j)^2\big]$$

satisfy the following estimate:

$$0 \leq \limsup_{t\to+\infty} E_{i,j}(t) \leq \left(\frac{\eta \left\| \lambda_i - \lambda_j \right|_\infty + \tilde{K}(n-2)(u_{\max} - u_{\min})}{\delta - 2nu_{\min}} \right)^2 . \tag{13.18}$$

Proof. We compute

$$\begin{aligned}
\frac{dE_{i,j}}{dt} &= (\dot{x}_i - \dot{x}_j)(x_i - x_j) + (\dot{y}_i - \dot{y}_j)(y_i - y_j) \\
&= \big[f_i(x_i, y_j) - f_j(x_j, y_j)\big](x_i - x_j) \\
&\quad + \left[-\sum_{k\in\mathscr{N}_i} u_{i,k}(t)(x_i - x_k) + \sum_{k\in\mathscr{N}_j} u_{j,k}(t)(x_j - x_k) \right](x_i - x_j) \\
&\quad + \big[g_i(x_i, y_j) - g_j(x_j, y_j)\big](y_i - y_j) \\
&\quad + \left[-\sum_{k\in\mathscr{N}_i} u_{i,k}(t)(y_i - y_k) + \sum_{k\in\mathscr{N}_j} u_{j,k}(t)(y_j - y_k) \right](y_i - y_j).
\end{aligned}$$

Next, we write

$$\sum_{k\in\mathscr{N}_i} u_{i,k}(t)(x_i-x_k) - \sum_{k\in\mathscr{N}_j} u_{j,k}(t)(x_j-x_k) = u_{i,j}(x_i-x_j) - u_{j,i}(x_j-x_i) + \sum_{k\in\mathscr{N}_i\setminus\{j\}} u_{i,k}(t)(x_i-x_k) - \sum_{k\in\mathscr{N}_j\setminus\{i\}} u_{j,k}(t)(x_j-x_k).$$

If the graph if complete, then we have $\mathscr{N}_i\setminus\{j\} = \mathscr{N}_j\setminus\{i\}$. Moreover, we have $u_{i,j}=u_{j,i}$. We obtain

$$\sum_{k\in\mathscr{N}_i} u_{i,k}(t)(x_i-x_k) - \sum_{k\in\mathscr{N}_j} u_{j,k}(t)(x_j-x_k) = 2u_{i,j}(x_i-x_j) + \sum_{k\in\mathscr{N}_i\setminus\{j\}} u_{i,k}(t)(x_i-x_k) - \sum_{k\in\mathscr{N}_j\setminus\{i\}} u_{j,k}(t)(x_j-x_k)$$

We introduce $\mathscr{S}_{i,j} = \mathscr{N}_i\setminus\{j\} = \mathscr{N}_j\setminus\{i\}$ and we observe that

$$\sum_{k\in\mathscr{S}_{i,j}} u_{i,k}(x_i-x_k) - \sum_{k\in\mathscr{S}_{i,j}} u_{j,k}(x_j-x_k) = \sum_{k\in\mathscr{S}_{i,j}} (u_{i,k}x_i - u_{j,k}x_j) - \sum_{k\in\mathscr{S}_{i,j}} (u_{i,k} - u_{j,k}x_k).$$

We write

$$\begin{aligned}(u_{i,k}x_i - u_{j,k}x_j)(x_i-x_j) &= (u_{i,k}x_i - u_{i,k}x_j)(x_i-x_j) + (u_{i,k}x_j - u_{j,k}x_j)(x_i-x_j)\\ &= u_{i,k}(x_i-x_j)^2 + x_j(u_{i,k}-u_{j,k})(x_i-x_j)\\ &\geq u_{\min}(x_i-x_j)^2 + x_j(u_{i,k}-u_{j,k})(x_i-x_j).\end{aligned}$$

Similarly, we have

$$(u_{i,k}x_i - u_{j,k}x_j)(x_i-x_j) \geq u_{\min}(x_i-x_j)^2 + x_i(u_{i,k}-u_{j,k})(x_i-x_j).$$

It follows that

$$(u_{i,k}x_i - u_{j,k}x_j)(x_i-x_j) \geq u_{\min}(x_i-x_j)^2 + \frac{x_i+x_j}{2}(u_{i,k}-u_{j,k})(x_i-x_j).$$

We can deduce

$$(x_i-x_j)\sum_{k\in\mathscr{S}_{i,j}} (u_{i,k}x_i - u_{j,k}x_j) \geq u_{\min}(n-2)(x_i-x_j)^2 + \frac{(x_i+x_j)(x_i-x_j)}{2}\sum_{k\in\mathscr{S}_{i,j}}(u_{i,k}-u_{j,k}),$$

since the set $\mathscr{S}_{i,j}$ contains $(n-2)$ elements. We obtain

$$\begin{aligned}&-(x_i-x_j)\Big[\sum_{k\in\mathscr{S}_{i,j}} u_{i,k}(x_i-x_k) - \sum_{k\in\mathscr{S}_{i,j}} u_{j,k}(x_j-x_k)\Big]\\ &\quad\leq -2u_{\min}(n-2)E^1_{i,j}\\ &\quad\quad - \frac{(x_i+x_j)(x_i-x_j)}{2}\sum_{k\in\mathscr{S}_{i,j}}(u_{i,k}-u_{j,k})\\ &\quad\quad + (x_i-x_j)\sum_{k\in\mathscr{S}_{i,j}}(u_{i,k}-u_{j,k})x_k,\end{aligned}$$

with $E^1_{i,j} = \frac{1}{2}(x_i - x_j)^2$. Next, we estimate

$$
\begin{aligned}
-\frac{(x_i+x_j)(x_i-x_j)}{2} & \sum_{k\in\mathscr{S}_{i,j}} (u_{i,k}-u_{j,k}) + (x_i-x_j)\sum_{k\in\mathscr{S}_{i,j}} (u_{i,k}-u_{j,k})x_k \\
& \leq \left|\frac{x_i+x_j}{2}\right| |x_i-x_j| \sum_{k\in\mathscr{S}_{i,j}} |u_{i,k}-u_{j,k}| \\
& + |x_i-x_j| \sum_{k\in\mathscr{S}_{i,j}} |u_{i,k}-u_{j,k}|\,|x_k| \\
& \leq K\,|x_i-x_j| \sum_{k\in\mathscr{S}_{i,j}} (u_{\max}-u_{\min}) \\
& + |x_i-x_j| \sum_{k\in\mathscr{S}_{i,j}} (u_{\max}-u_{\min})K,
\end{aligned}
$$

where K is a positive constant such that $|x_i| \leq K$ for all i, whose existence is guaranteed by Theorem 3. We obtain

$$
\begin{aligned}
-(x_i-x_j)\Big[&\sum_{k\in\mathscr{S}_{i,j}} u_{i,k}(x_i-x_k) - \sum_{k\in\mathscr{S}_{i,j}} u_{j,k}(x_j-x_k)\Big] \\
&\leq -2u_{\min}(n-2)E^1_{i,j} + 2K(n-2)\,|x_i-x_j|\,(u_{\max}-u_{\min}).
\end{aligned}
$$

Similarly, we estimate

$$
\begin{aligned}
-(y_i-y_j)\Big[&\sum_{k\in\mathscr{S}_{i,j}} u_{i,k}(y_i-y_k) - \sum_{k\in\mathscr{S}_{i,j}} u_{j,k}(y_j-y_k)\Big] \\
&\leq -2u_{\min}(n-2)E^2_{i,j} + 2K(n-2)\,|y_i-y_j|\,(u_{\max}-u_{\min}),
\end{aligned}
$$

where $E^2_{i,j} = \frac{1}{2}(y_i - x_j)^2$. Now we come back to estimate $\frac{dE_{i,j}}{dt}$:

$$
\begin{aligned}
\frac{dE_{i,j}}{dt} &\leq \big[f_i(x_i,y_i) - f_j(x_j,y_j)\big](x_i-x_j) + \big[g_i(x_i,y_i) - g_j(x_j,y_j)\big](y_i-y_j) \\
&- 2u_{\min}nE_{i,j} + 2K(n-2)\big[\,|x_i-x_j| + |y_i-y_j|\,\big] \\
&\leq \big[\eta\,\|\lambda_i-\lambda_j|_\infty + \tilde{K}(n-2)(u_{\max}-u_{\min})\big]E^{1/2}_{i,j} + \big[\delta - 2nu_{\min}\big]E_{i,j}.
\end{aligned}
$$

Applying Gronwall Lemma and comparing with the solution of the Bernoulli equation (13.13) finishes the proof.

Corollary 4. *The controlled system* (13.15) *nearly synchronizes with respect to* $u_{\min}$*, provided* $(u_{\max} - u_{\min})$ *is uniformly bounded.*

Remark 2 (Generalization to other models). We can consider a complex network of ecological systems of the general form

$$\dot{x} = xM(x,y,\lambda), \qquad \dot{y} = yN(x,y,\lambda), \tag{13.19}$$

where M and N are regular functions defined in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^p$ (see notably [7], [28]), by considering the following system:

$$\begin{cases} \dot{x}_i = xM_i(x_i, y_i) - \sum_{j \in \mathcal{N}_i} u_{i,j}(x_i - x_j), \\ \dot{y}_i = yN_i(x_i, y_i) - \sum_{j \in \mathcal{N}_i} u_{i,j}(y_i - y_j), \end{cases} \tag{13.20}$$

with $1 \leq i \leq n$ and control functions $u_{i,j}$. Here, the functions M_i and N_i are non identical instances of the functions M, N defined in (13.19), that is

$$M_i(x_i, y_i) = M(x_i, y_i, \lambda_i), \qquad N_i(x_i, y_i) = N(x_i, y_i, \lambda_i),$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}^p$. Theorem 2 and its corollaries can easily be generalized to this setting, with a complex network of the form (13.20), under the single assumption that there exists constants $\eta > 0$ and $\delta > 0$ such that

$$\begin{pmatrix} f_i(x_i, y_i) - f_j(x_j, y_j) \\ g_i(x_i, y_i) - g_j(x_j, y_j) \end{pmatrix} \cdot \begin{pmatrix} x_i - x_j \\ y_i - y_j \end{pmatrix} \leq \eta \left\|\lambda_i - \lambda_j\right|_\infty E_{i,j}^{1/2} + \delta E_{i,j}. \tag{13.21}$$

Remark 3 (Non complete graph topologies). Recent results have been obtained (see [2]) for synchronization in non complete graph topologies.

The previous result motivates the setting of an optimal control problem, so as to exert a command on the dynamics of the complex network (13.9) and to reach a synchronization state, even in the case of non-identical patches.

13.4 Optimal Control Synchronization

Considering the controlled complex network of Lotka-Volterra systems (13.15), we propose an optimal control problem, in order to exert a command on the global behavior of this complex network. To model the goal of restoring biodiversity and biological dynamics in a fragmented environment, we need define an appropriate cost functional.

The choice of the cost functional has an important role on the optimal synchronization of the complex network (see e.g. [25] for a study on the role of the objective functional in optimal control problems applied to compartmental models for biomedical therapies). Here, our main focus on the choice of the cost functional is not only on the properties of the optimal control solution but mainly on the dynamics of the state functions x_i, y_i ensuring the conservation of both species.

In what follows, we will consider different cost functionals where the conservation of species is guaranteed by:

i. imposing synchronization;
ii. synchronization of limit cycles.

Impose synchronization – optimal solutions may kill *limit cycles or damped oscillations*

For the scenario "*imposing synchronization*", we consider the optimal control problem of determining $X^*(\cdot)$ associated to the admissible controls $u_{i,j}^*(\cdot) \in \Omega$ on the time interval $[0, T]$, satisfying the controlled system $\dot{X} = F(X, \Lambda) + G(X, \{u_{i,j}\}_{1 \leq i,j \leq n})$, given by (13.15), the fixed initial conditions $X(0) = X_0 \in (\mathbb{R}^+)^{2n}$ and minimizing the one of the following cost functionals:

$$J_1 = \int_0^T \sum_{i \neq j} \left[(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2 \right] dt. \tag{13.22}$$

or

$$J_2 = \sum_{i \neq j} \left[(x_i(T) - x_j(T))^2 + (y_i(T) - y_j(T))^2 \right]. \tag{13.23}$$

In this case, we emphasize that maximizing synchronization can *kill* limit cycles or damped oscillations possibly occurring in the case of constant couplings (σ_1, σ_2). Indeed, let us consider as a simple example a four nodes network, associated with a complete graph topology (as shown in Figure 13.1). The parameters of each patch are given in Table 13.1, and the initial conditions were randomly generated between 0 and 1. If the couplings are fixed to $\sigma_1 = \sigma_2 = 1$, then the local dynamics are synchronized towards the same damped oscillations (when $\alpha_i = 0.4$, $1 \leq i \leq 4$, see Figure 13.2) or towards the same limit cycle (when $\alpha_i = 0.3$, $1 \leq i \leq 4$, see Figure 13.3). However, when the couplings are determined by a control associated with the functionals J_1, J_2 given by (13.22), (13.23), then we observe that the oscillations vanish, as depicted in Figure 13.4. In other words, the optimality criterion can lead to an *unexpected* emergent behavior.

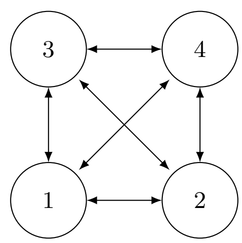


Fig. 13.1: Simple 4 nodes complex network with a complete graph topology.

Table 13.1: Values of the parameters for the example of a 4 nodes network.

Parameter	**Value**	**Parameter**	**Value**
r_1	0.8	r_3	0.9
d_1	0.98	d_3	0.7
c_1	1.6	c_3	1.6
α_1	0.3, 0.4	α_3	0.3, 0.4
r_2	0.8	r_4	0.8
d_2	0.6	d_4	0.75
c_2	1.6	c_4	1.6
α_2	0.3, 0.4	α_4	0.3, 0.4

Remark 4. The non nonexistence of a limit cycle in an optimal control problem applied to a diabetes model was proved in [9].

This lead us to consider an alternative cost functional for which the optimal solution is attracted to a local limit cycle.

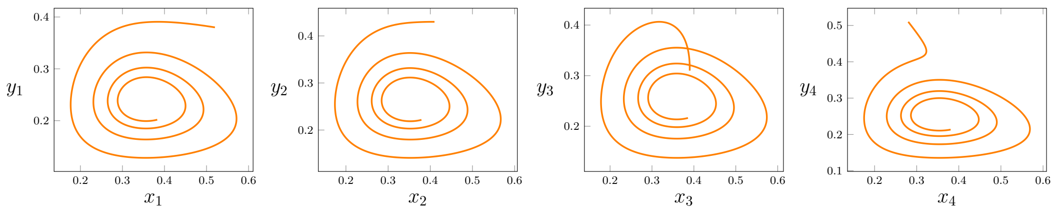


Fig. 13.2: Synchronization of damped oscillations in a four nodes network with constant couplings. Even if the initial conditions and parameters are distinct from one patch to another, the local dynamics are synchronized towards the same damped oscillations.

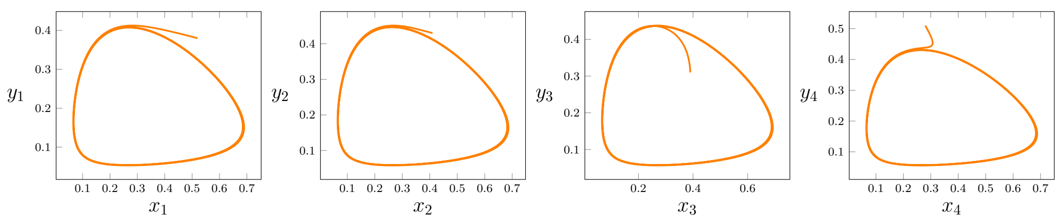


Fig. 13.3: Synchronization of limit cycles in a four nodes network with constant couplings.

How to reach a limit cycle?

Suppose that without couplings, the local dynamics are attracted to local limit cycles. Moreover, suppose that we can synchronize (or near synchronize) these local limit cycles with a constant coupling strength.

We denote by $\gamma(t) = \big(\zeta(t), \xi(t)\big)_{0 \leq t \leq \phi}$ a parametrization of the global limit cycle (with period ϕ), obtained by synchronization with a constant coupling strength.

Then we can try to preserve and reach this cycle in an optimal control problem by minimizing the cost functional

$$J_\gamma\left(x_i, y_i\right) = \sum_{k=0}^{k^*} \int_{T+k\phi}^{T+(k+1)\phi} \sum_{i=1}^{n} \Big[\left(x_i(t) - \zeta(t)\right)^2 + \left(y_i(t) - \xi(t)\right)^2 \Big] \, dt \,, \tag{13.24}$$

where T is a positive time such that the transitional dynamics occurs in $[0, T]$, and k^* is the number of periods of oscillations around the limit cycles. Considering again the four nodes network given in Figure 13.1 and the parameters given in Table 13.1, we show in Figure 13.5 a first simulation of the control problem determined by the functional (13.24). We observe that controls can be found to maintain oscillations. It is now our aim to analyze the properties of these controls in a rigorous framework.

13.4.1 Optimal control problem

Our goal is to find the optimal control solution that reaches a limit cycle γ, ensuring the preservation of the ecological biodiversity in a fragmented environment. We consider the optimal control problem

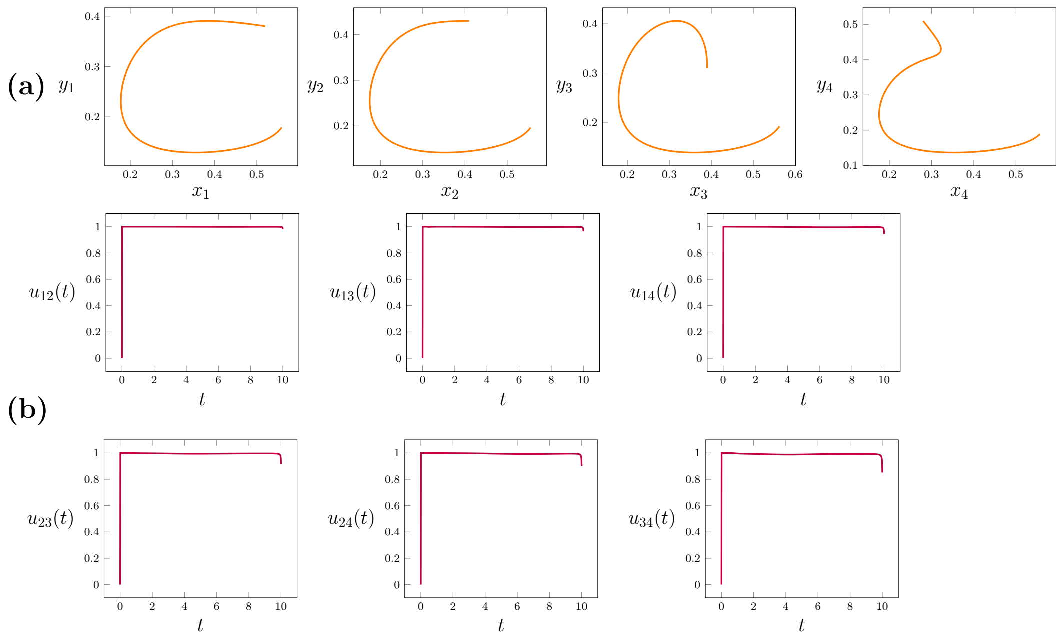


Fig. 13.4: Imposing synchronization in a four nodes network with a control associated with the functionals (13.22), (13.23) can kill oscillations or limit cycles. **(a)** Phase portraits showing the local dynamics (x_i, y_i) on each patch i of the network $(1 \leq i \leq 4)$. **(b)** Time series of the control functions u_{ij} between each pair (i, j) of patches $(1 \leq i, j \leq j,\ i \neq j)$.

(OCP) given by

$$\min_{X,u} \sum_{k=0}^{k^*} \int_{T+k\phi}^{T+(k+1)\phi} \sum_{i=1}^{n} \left[\left(x_i(t) - \zeta(t)\right)^2 + \left(y_i(t) - \xi(t)\right)^2 \right] dt\,,$$

such that

$$\begin{cases} \dot{x}_i = r_i x_i(1 - x_i) - \dfrac{c_i x_i y_i}{\alpha_i + x_i} - \displaystyle\sum_{j \in \mathcal{N}_i} u_{i,j}(x_i - x_j), \\ \dot{y}_i = -d_i y_i + \dfrac{c_i x_i y_i}{\alpha_i + x_i} - \displaystyle\sum_{j \in \mathcal{N}_i} u_{i,j}(y_i - y_j), \end{cases} \qquad \text{(OCP)}$$

with $X^*(0) = X_0^* \in (\mathbb{R}^+)^{2n}$ and $u_{i,j}(\cdot) \in \Omega\,.$

The existence of solutions for the (OCP) is ensured by classical sufficient conditions, see e.g. [34] and references cited therein.

Let us now apply the well known first order necessary optimality condition given by the Pontryagin maximum principle [30] to the (OCP) problem. In what follows, we write (x_i, y_i) for $(x_i(t), y_i(t)) \in (\mathbb{R}^+)^{2n}$, $u_{i,j}$ for $u_{i,j}(t) \in \Omega$ and $p = (p_{1,i}, p_{2,i})$ for $(p_{1,i}(t), p_{2,i}(t)) : [0, T] \to (\mathbb{R}^+)^{2n}$,

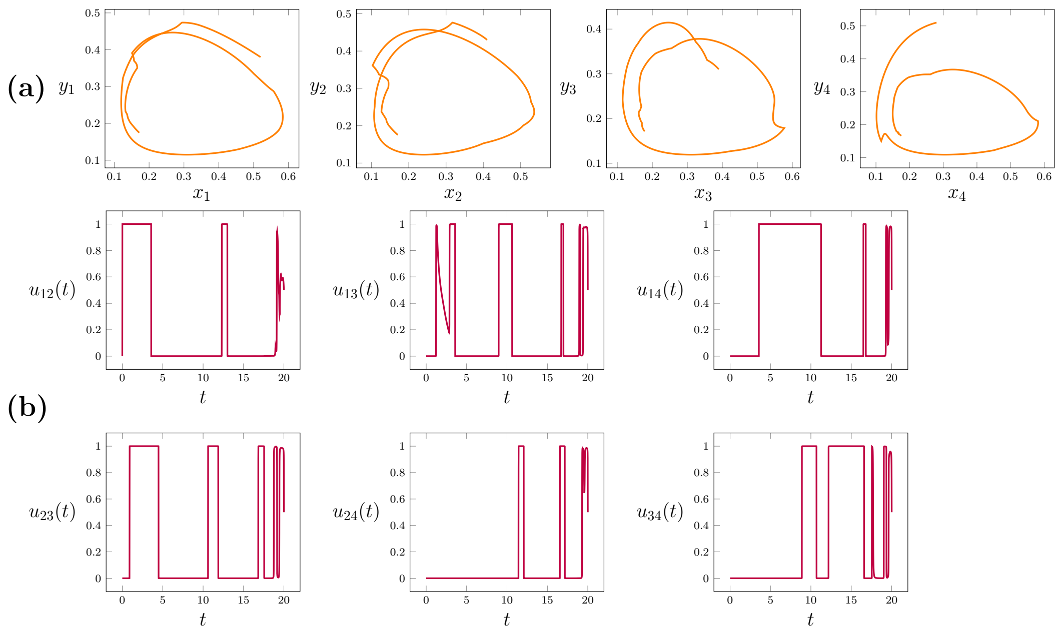


Fig. 13.5: Attempt to reach synchronization of a given limit cycle in a four nodes network. **(a)** Phase portraits showing the local dynamics (x_i, y_i) on each patch i of the network $(1 \leq i \leq 4)$. **(b)** Time series of the control functions u_{ij} between each pair (i, j) of patches $(1 \leq i, j \leq j,\, i \neq j)$.

with $t \in [0, T]$ and $1 \leq i \leq n$. According to the Pontryagin maximum principle, if $u^*_{i,j} \in \Omega$ is optimal for (OCP), then there exists a nontrivial absolutely continuous mapping p^*, the adjoint vector, such that

$$\dot{x}^*_i = \frac{\partial H}{\partial p^*_{1,i}}\,, \quad \dot{y}^*_i = \frac{\partial H}{\partial p^*_{2,i}}\,,$$

and

$$\dot{p^*}_{1,i} = -\frac{\partial H}{\partial x^*_i} \quad \text{and} \quad \dot{p^*}_{2,i} = -\frac{\partial H}{\partial y^*_i} \text{ for } 1 \leq i \leq n\,,$$

where the normalized *Hamiltonian* H is given by

$$\begin{aligned} H\left(x_i, y_i, p_{1,i}, p_{2,i}, u_{i,j}\right) = & -\sum_{i=1}^{n} \Big[\left(x_i(t) - \zeta(t)\right)^2 + \left(y_i(t) - \xi(t)\right)^2 \Big] \\ & + \sum_{i=1}^{n} p_{1,i}(t) \left(f_1\left(x_i, y_i, \lambda_i\right) + g_1\left(x_i, X, u_{i,j}\right)\right) \\ & + \sum_{i=1}^{n} p_{2,i}(t) \left(f_2\left(x_i, y_i, \lambda_i\right) + g_2\left(y_i, X, u_{i,j}\right)\right)\,, \end{aligned} \tag{13.25}$$

for $1 \leq i \leq n$ and $j \in \mathcal{N}_i$. The minimization condition is given by

$$H\left(x_i^*, y_i^*, p_{1,i}^*, p_{2,i}^*, u_{i,j}^*\right) = \min_{u_{i,j}\in\Omega} H\left(x_i^*, y_i^*, p_{1,i}^*, p_{2,i}^*, u_{i,j}\right)\,,$$

holds almost everywhere on $[0,T]$. Moreover, the *transversality conditions* $\left(p_{1,i}^*(T), p_{2,i}^*(T)\right) = (0,0)$, hold, with $1 \le i \le n$.

The minimizing controls $u_{i,j}^*$ are determined by the *switching functions*

$$\phi_{i,j} = \frac{\partial H}{\partial u_{i,j}}\,, \text{for } 1 \le i \le n \text{ and } j \in \mathscr{N}_i$$

and the *control law*

$$u_{i,j}^*(t) = \begin{cases} 0 & \text{if} \quad \phi_{i,j}(t) > 0\,, \\ u_{max} & \text{if} \quad \phi_{i,j}(t) < 0\,, \\ \text{singular} & \text{if} \quad \phi_{i,j}(t) = 0 \quad \forall t \in I_s \subset [0,T]\,, \end{cases} \tag{13.26}$$

for $1 \le i \le n$ and $j \in \mathscr{N}_i$.

If the switching functions $\phi_{i,j}$, $1 \le i \le n$, $j \in \mathscr{N}_i$, do not vanish on any subinterval I of $[0,T]$, then the extremal controls $u_{i,j}^*$ are *bang-bang* on I. The zeros of $\phi_{i,j}$ on I (possibly in infinite number), $0 < \tau_1^* < \ldots < \tau_s^* \ldots$, are called the *switching times*.

Moroever, the *strict bang-bang Legendre condition*, can be applied to the (OCP) problem, that is,

$$\dot{\phi}_{i,j}(\tau_l^*) = \frac{d}{dt}\phi(t)|_{t=\tau_l^*} \neq 0\,, \tag{13.27}$$

for every switching time.

Next, we consider an optimal control synchronization problem of the type (OCP) with 4 nodes.

13.4.2 Example: optimal control synchronization with 4 nodes

Let γ be limit cycle, that is, $\{\gamma(t) = (\zeta(t), \xi(t))\}_{0\le t\le\phi}$, with period ϕ.

Consider the control system with 4 nodes, again associated with the complete graph shown in Figure 13.1. The equations of the controlled network are given by

$$\begin{cases} \dot{x}_1 = r_1x_1(1-x_1) - \frac{c_1x_1y_1}{\alpha_1+x_1} - \sum\limits_{j\neq 1} u_{1,j}(x_1-x_j) \\ \dot{y}_1 = -d_1y_1 + \frac{c_1x_1y_1}{\alpha_1+x_1} - \sum\limits_{j\neq 1} u_{2,j}(y_1-y_j) \\ \dot{x}_2 = r_2x_2(1-x_2) - \frac{c_2x_2y_2}{\alpha_2+x_2} - \sum\limits_{j\neq 2} u_{2,j}(x_2-x_j) \\ \dot{y}_2 = -d_2y_2 + \frac{c_2x_2y_2}{\alpha_2+x_2} - \sum\limits_{j\neq 2} u_{1,j}(y_2-y_j) \\ \dot{x}_3 = r_3x_3(1-x_3) - \frac{c_3x_3y_3}{\alpha_3+x_3} - \sum\limits_{j\neq 3} u_{3,j}(x_3-x_j) \\ \dot{y}_3 = -d_3y_3 + \frac{c_3x_3y_3}{\alpha_3+x_3} - \sum\limits_{j\neq 3} u_{3,j}(y_3-y_j) \\ \dot{x}_4 = r_4x_4(1-x_4) - \frac{c_4x_4y_4}{\alpha_4+x_4} - \sum\limits_{j\neq 4} u_{4,j}(x_4-x_j) \\ \dot{y}_4 = -d_4y_4 + \frac{c_4x_4y_4}{\alpha_4+x_4} - \sum\limits_{j\neq 4} u_{4,j}(y_4-y_j) \end{cases} \tag{13.28}$$

and the cost functional

$$J\left(x_i, y_i\right) = \sum_{k=0}^{k^*} \int_{T+k\phi}^{T+(k+1)\phi} \sum_{i=1}^{4} \left[\left(x_i(t) - \zeta(t)\right)^2 + \left(y_i(t) - \xi(t)\right)^2 \right] \, dt, \tag{13.29}$$

subject to the initial conditions $X(0) = (x_1(0), \ldots, x_4(0)) \in \mathcal{N}(\gamma)$, and control constraints

$$u_{min} \leq u_{i,j}(t) \leq u_{max} \, , \quad t \in [T, T + m\phi],$$

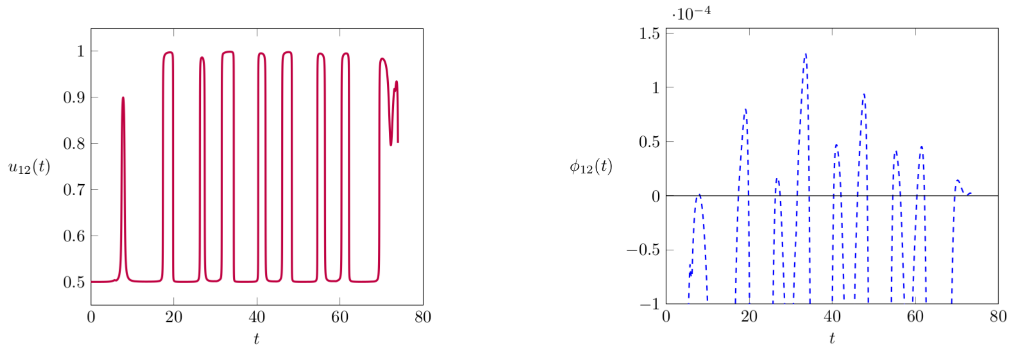
where m is the number of periods and u_{min} satisfies $u_{min} \geq K$, with K a positive threshold, that guarantees the near synchronization in the case of a constant coupling strength σ. The normalized Hamiltonian is given by

$$\begin{aligned}
H = & - \sum_{i=1}^{4} \left[\left(x_i(t) - \zeta(t)\right)^2 + \left(y_i(t) - \xi(t)\right)^2 \right] \\
& + p_{1,1} \left(r_1 x_1 (1 - x_1) - \frac{c_1 x_1 y_1}{\alpha_1 + x_1} - u_{1,2}(x_1 - x_2) - u_{1,3}(x_1 - x_3) - u_{1,4}(x_1 - x_4) \right) \\
& + p_{2,1} \left(-d_1 y_1 + \frac{c_1 x_1 y_1}{\alpha_1 + x_1} - u_{1,2}(y_1 - y_2) - u_{1,3}(y_1 - y_3) - u_{1,4}(y_1 - y_4) \right) \\
& + p_{1,2} \left(r_2 x_2 (1 - x_2) - \frac{c_2 x_2 y_2}{\alpha_2 + x_2} - u_{1,2}(x_2 - x_1) - u_{2,3}(x_2 - x_3) - u_{2,4}(x_2 - x_4) \right) \\
& + p_{2,2} \left(-d_2 y_2 + \frac{c_2 x_2 y_2}{\alpha_2 + x_2} - u_{1,2}(y_2 - y_1) - u_{2,3}(y_2 - y_3) - u_{2,4}(y_2 - y_4) \right) \\
& + p_{1,3} \left(r_3 x_3 (1 - x_3) - \frac{c_3 x_3 y_3}{\alpha_3 + x_3} - u_{1,3}(x_3 - x_1) - u_{2,3}(x_3 - x_2) - u_{3,4}(x_3 - x_4) \right) \\
& + p_{2,3} \left(-d_3 y_3 + \frac{c_3 x_3 y_3}{\alpha_3 + x_3} - u_{1,3}(y_3 - y_1) - u_{2,3}(y_3 - y_2) - u_{3,4}(y_3 - y_4) \right) \\
& + p_{1,4} \left(r_4 x_4 (1 - x_4) - \frac{c_4 x_4 y_4}{\alpha_4 + x_4} - u_{1,4}(x_4 - x_1) - u_{2,4}(x_4 - x_2) - u_{3,4}(x_4 - x_3) \right) \\
& + p_{2,4} \left(-d_4 y_4 + \frac{c_4 x_4 y_4}{\alpha_4 + x_4} - u_{1,4}(y_4 - y_1) - u_{2,4}(y_4 - x_2) - u_{3,4}(y_4 - y_3) \right) \, ,
\end{aligned}$$

and the switching functions are given by

$$\begin{aligned}
\phi_{1,2} &= \frac{\partial H}{\partial u_{12}} = -p_{1,1}(x_1 - x_2) - p_{2,1}(y_1 - y_2) - p_{1,2}(x_2 - x_1) - p_{2,2}(y_2 - y_1) \, , \\
\phi_{1,3} &= \frac{\partial H}{\partial u_{13}} = -p_{1,1}(x_1 - x_3) - p_{2,1}(y_1 - y_3) - p_{1,3}(x_3 - x_1) - p_{2,3}(y_3 - y_1) \, , \\
\phi_{1,4} &= \frac{\partial H}{\partial u_{14}} = -p_{1,1}(x_1 - x_4) - p_{2,1}(y_1 - y_4) - p_{1,4}(x_4 - x_1) - p_{2,4}(y_4 - y_1) \, , \\
\phi_{2,3} &= \frac{\partial H}{\partial u_{23}} = -p_{1,2}(x_2 - x_3) - p_{2,2}(y_2 - y_3) - p_{1,3}(x_3 - x_2) - p_{2,3}(y_3 - y_2) \, , \\
\phi_{2,4} &= \frac{\partial H}{\partial u_{24}} = -p_{1,2}(x_2 - x_4) - p_{2,2}(y_2 - y_4) - p_{1,4}(x_4 - x_2) - p_{2,4}(y_4 - x_2) \, , \\
\phi_{3,4} &= \frac{\partial H}{\partial u_{34}} = -p_{1,3}(x_3 - x_4) - p_{2,3}(y_3 - y_4) - p_{1,4}(x_4 - x_3) - p_{2,4}(y_4 - y_3) \, .
\end{aligned}$$

In Figure 13.6 we observe that the control $u_{1,2}$ and the corresponding switching function $\phi_{1,2}$ satisfy the control law (13.26) and the strict bang-bang Legendre condition (13.27). Analogously, the other controls also satisfy these optimality conditions, but for simplicity we do not provide a figure for the others five controls.



(a) Control $u_{1,2}$.

(b) Switching function $\phi_{1,2}$.

Fig. 13.6: The control $u_{1,2}$ and the switching function $\phi_{1,2}$ satisfy (13.26) and (13.27).

Next, we show in Figure 13.7 the dynamics of the solution to the controlled four nodes network (13.28)-(13.29). We have computed the solution $\Big((x_i, y_i)_{1\leq i\leq 4}, (u_{i,j})_{1\leq i\neq j\leq 4}\Big)$ until the final time $T + m\phi$ with $T = 6.5$, $m = 5$ and $\phi = 13.5$. It is interesting to note that the numbers of switching times of the controls are distinct. Namely, u_{12} reaches 9 times its maximum value, whereas u_{23} does only 5 times. We mainly observe that oscillations are maintained under the action of bang-bang controls. Overall, our goal to synchronize the local dynamics while preserving oscillations is reached.

13.5 Conclusion and Future Work

In this chapter, we considered a controlled complex network of Lotka-Volterra systems, where the strength of the migrations of biological individuals between the patches is replaced by control functions, reproducing the implementation of ecological corridors in a fragmented environment. We assumed that the ecological dynamics are non-identical within the fragmented environment and proved near-synchronization sufficient conditions for the solution of the controlled complex network.

After, we study optimal control problems where the main goal is the minimization of the default of synchronization in the complex network. We consider different cost functionals taking into account that the dynamics of the controlled complex network ensure the conservation of both species, namely, our goal is to impose synchronization or synchronization of limit cycles. Therefore, the solutions of the optimal control problems lead to a restoration of the biodiversity of life species in a heterogeneous habitat by reaching at least a global coexistence equilibrium, or in a better scenario, a global limit cycle which would guarantee biological oscillations, which means rich life dynamics.

In a future work, we aim to enlarge our study of controlled synchronization or near-synchronization in complex networks of nonlinear dynamical systems. First, it is natural to ask

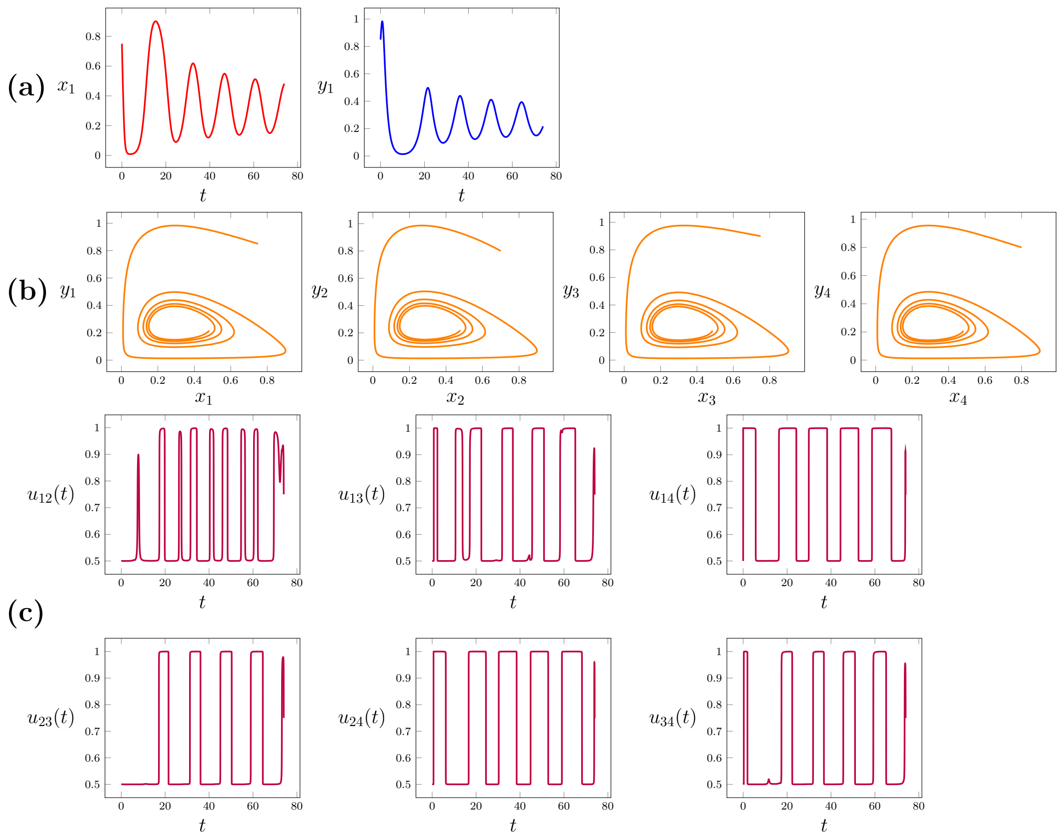


Fig. 13.7: Synchronization towards oscillations of the controlled four nodes network (13.28)–(13.29). **(a)** Time series showing the evolution of x_1, y_1 on the first patch. **(b)** Phase portraits showing the local dynamics (x_i, y_i) on each patch i of the network $(1 \leq i \leq 4)$. **(c)** Time series of the control functions u_{ij} between each pair (i, j) of patches $(1 \leq i, j \leq j,\ i \neq j)$.

if the possibility to near-synchronize oscillations in finite-dimensional systems can be generalized to infinite dimensional systems, such as reaction-diffusion systems, which are likely to admit bifurcations of periodic solutions (see for instance [26] for a study of oscillatory solutions in a spatial Holling-Tanner reaction-diffusion system). Next, another exciting perspective would be to investigate the optimal control of synchronization of chaotic systems, since it is known that such systems can be synchronized by constant couplings (see notably [1]). Hence, we believe that optimal control of synchronization in complex networks of nonlinear dynamical systems will produce original results in a near future.

Acknowledgments

This work is partially supported by Portuguese funds through CIDMA, The Center for Research and Development in Mathematics and Applications of University of Aveiro, and the Portuguese Foundation for Science and Technology (FCT – Fundação para a Ciência e a Tecnologia), within project UIDB/04106/2020 (https://doi.org/10.54499/UIDB/04106/2020) and by the project "Mathematical Modelling of Multi-scale Control Systems: applications to human diseases (CoSysM3)", 2022.03091.PTDC, financially supported by national funds (OE), through FCT/MCTES.

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