QUATERNIONIC ESSENTIAL NUMERICAL RANGE OF COMPLEX OPERATORS

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Resumo. We study the essential numerical range of complex operators on a quaternionic Hilbert space and its relation with the essential S-spectrum. We give a new characterization of the essential numerical range relating it to the complex essential numerical range. Moreover, we show that the quaternionic essential numerical range of a normal operator is the convex hull of the essential S-spectrum.

Introduction

Let $\mathbb{F}$ be the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or the skew field $\mathbb{H}$ of Hamilton quaternions. Let $\mathcal{H}$ be a right Hilbert space over $\mathbb{F}$ and let T be a bounded linear operator on $\mathcal{H}$. The numerical range of T is the set

$$W_{\mathbb{F}}(T) = \{\langle Tx, x\rangle : \|x\| = 1, x \in \mathcal{H}\},$$

where $\langle \cdot\,, \cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbb{F}$ is the inner product on $\mathcal{H}$. It is well known that the numerical range is a bounded (not necessarily closed) subset of $\mathbb{F}$ and that convexity of $W_{\mathbb{F}}(T)$ depends on the ground field $\mathbb{F}$. In fact, when $\mathcal{H}$ is a real or a complex Hilbert space, $W_{\mathbb{F}}(T)$ is always a convex set, as stated by the Toeplitz Hausdorff Theorem (see [GR]), whereas in the quaternionic setting the convexity is not guaranteed (for more details on quaternionic numerical range see our papers on the subject [CDM1], [CDM2], [CDM3],[CDM4],[CDM5],[CDM6] and also [K], [MBB], [Ye]).

The closure of the numerical range of T contains a non-empty compact subset called the essential numerical range, defined by

$$W_{e,\mathbb{F}}(T) = \bigcap_{K\in\mathcal{K}(\mathcal{H})} \overline{W_{\mathbb{F}}(T + K)}, \tag{0.1}$$

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where $\mathcal{K}(\mathcal{H})$ denotes the set of compact operators on the $\mathbb{F}$-Hilbert space $\mathcal{H}$. Throughout the paper we let $W(T) = W_{\mathbb{H}}(T)$ denote the quaternionic numerical range of T and $W_e(T) = W_{e,\mathbb{H}}(T)$ the quaternionic essential numerical range of T.

The notion of complex essential numerical range was introduced in [SW]. Later, in [FSW], several new characterizations were obtained. A natural extension to the quaternionic setting was given in [CDMS1]. Interestingly, it was proved in that paper that despite $W(T)$ is not convex in general, $W_e(T)$ is always a convex set.

This work is concerned with the relationship between the quaternionic and the complex essential numerical ranges for complex operators on a quaternionic Hilbert space. For this matter, we introduce the rather natural notion of essential bild. In fact, since $q \in W_e(T)$ if and only if $[q] \subseteq W_e(T)$, see [CDMS1], it is enough to study the subset of complex elements in each similarity class, that is

$$\begin{aligned} B_e(T) &= W_e(T) \cap \mathbb{C} \\ &= \bigcap_{K \in \mathcal{K}(\mathcal{H})} \overline{B(T+K)}, \end{aligned}$$

where $B(A) = W(A) \cap \mathbb{C}$ denotes the bild of an operator $A \in \mathcal{B}(\mathcal{H})$. The set $B_e(T)$ is called the essential bild of $T \in \mathcal{B}(\mathcal{H})$ and it is non-empty, closed and convex.

The article is organised as follows. We start Section 1 by recalling a few results about the essential numerical range in the quaternionic setting. In Section 2 we recall complex and real operators on a quaternionic Hilbert space (see [CDM6, Definition 2.4]) and we give a new characterization of the essential numerical range of these two classes of operators (see Theorem 2.3). Each class (real and complex) determines the class of compact operators necessary to build up the essential numerical range. Namely, we show that to compute the quaternionic essential numerical range of a complex operator, the intersection in (0.1) can be taken over the complex compact operators in place of quaternionic ones. For real operators, it is enough to intersect over the real compact operators to obtain both the quaternionic and the complex essential numerical ranges. This characterization allows us to compute the essential bild of an operator T, and therefore infer about $W_e(T)$, in terms of the complex essential numerical of T and T^*, see Theorem 2.4. It follows that when T is a real operator, $B_e(T) = W_{e,\mathbb{C}}(T)$, see Corollary 2.5. In Section 3 we show that the essential S-spectrum (definition in Section 1) of a complex operator is given by the similarity class of its essential $\mathbb{C}$-spectrum, see Propostition 3.2. From this and the invariance under approximately unitary equivalence of the essential S-spectrum and the essential numerical range (Proposition 3.4), we show that the quaternionic essential numerical range of a normal operator is the convex hull of the essential S-spectrum (Theorem 3.5).

1. Preliminaries

The division ring of real quaternions $\mathbb{H}$, also known as Hamilton quaternions, is an algebra over the field of real numbers with basis $\{1, i, j, k\}$ and product defined by $i^2 = j^2 = k^2 = ijk = -1$. Given a quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$, its conjugate is $q^* = q_0 - q_1 i - q_2 j - q_3 k$. We call $Re(q) = \frac{q+q^*}{2}$ and $Im(q) = \frac{q-q^*}{2}$ the real and imaginary parts of q, respectively. The norm of q is the nonnegative real number $|q| = \sqrt{qq^*}$. Two quaternions $q, q' \in \mathbb{H}$ are similar if there is a unit $u \in \mathbb{H}$ such that $u^* q u = q'$, in which case we write $q \sim q'$. This is an equivalence relation and we denote the equivalence class of q by $[q]$.

Let $\mathcal{H}$ denote a right Hilbert space over $\mathbb{H}$. In particular, the norm of $x \in \mathcal{H}$ is defined by the underlying $\mathbb{H}$-inner product as $\|x\| = \sqrt{\langle x, x\rangle}$. The space of bounded, right $\mathbb{H}$-linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$, its closed ideal of compact operators by $\mathcal{K}(\mathcal{H})$ and the group of invertible operators by $\mathcal{B}(\mathcal{H})^{-1}$.

For an operator $T \in \mathcal{B}(\mathcal{H})$, let $(rT)x = (Tx)r$, for $x \in \mathcal{H}$ and $r \in \mathbb{R}$. Furthermore, for $q \in \mathbb{H}$, we define $\Delta_q(T) : \mathcal{H} \to \mathcal{H}$ by

$$\Delta_q(T) = T^2 - 2Re(q)T + |q|^2 I,$$

where I is the identity operator. Clearly, $\Delta_q(T) \in \mathcal{B}(\mathcal{H})$. The S-spectrum of T, see [CGSS], is defined by

$$\sigma_S(T) = \{q \in \mathbb{H} : \Delta_q(T) \notin \mathcal{B}(\mathcal{H})^{-1}\}.$$

Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denote the canonical quotient map and $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ the Calkin algebra. Let $\pi(T) = [T]$ denote the equivalence class $T + \mathcal{K}(\mathcal{H})$, for $T \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{C}(\mathcal{H})$ is a normed algebra with $\|[T]\| = \inf_{K \in \mathcal{K}(\mathcal{H})} \|T + K\| \leqslant \|T\|$. We say that T is a Fredholm operator if the class $[T]$ is invertible in $\mathcal{C}(\mathcal{H})$. According to Atkinson Theorem, $T \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator if and only if its range is closed and the kernels $\ker(T)$ and $\ker(T^*)$ are finite dimensional, where $T^* \in \mathcal{B}(\mathcal{H})$ is the adjoint of T. The set of all Fredholm operators in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{F}(\mathcal{H})$.

The essential S-spectrum of $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$\sigma_e^S(T) = \{q \in \mathbb{H} : \Delta_q(T) \notin \mathcal{F}(\mathcal{H})\}. \tag{1.1}$$

This is a non-empty compact subset of $\sigma_S(T)$, see [MT]. One can show that $\sigma_e^S(T)$ is circular, that is, $q \in \sigma_e^S(T)$ is equivalent to $[q] \subseteq \sigma_e^S(T)$.

In [CDMS1], we defined the quaternionic essential numerical range of $T \in \mathcal{B}(\mathcal{H})$ as the set

$$W_e(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \overline{W(T+K)}.$$

The essential numerical range is a non-empty, compact and convex subset of $\mathbb{H}$, contained in the closed disc $\overline{\mathbb{D}(0, \|\pi(T)\|)}$. Clearly, $W_e(T) \subseteq \overline{W(T)}$. Moreover, $\sigma_e^S(T) \subseteq W_e(T)$. Another important property of $W_e(T)$ is that it is invariant under unitary equivalence, that is, $W_e(UTU^*) = W_e(T)$, for every unitary operator $U \in \mathcal{B}(\mathcal{H})$. See [CDMS1] for further details.

Alternative characterizations of the quaternionic essential numerical range were obtained in [CDMS1, Theorem 2.2].

Theorem 1.1 ([CDMS1])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in \mathbb{H}$. The following conditions are equivalent:*

(i) $q \in W_e(T)$.

(ii) There exists a sequence $(x_n)_n$ of unit vectors in $\mathcal{H}$ such that $x_n \rightharpoonup 0$ and $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = q$.

(iii) There exists an orthonormal sequence $(e_n)_n$ in $\mathcal{H}$ such that $\lim_{n\to\infty}\langle Te_n, e_n\rangle = q$.

A sequence satisfying condition (ii) is called an essential sequence for q, see [CDMS1, Definition 2.3].

2. Complex operators

In this section we introduce the notion of complex operators on $\mathcal{H}$, where $\mathcal{H}$ is a quaternionic Hilbert space, as defined in [CDM6]. This class of operators enjoys some interesting properties allowing us to give a more detailed characterization of their quaternionic essential numerical range.

For every orthonormal basis $\mathcal{E} = \{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$, there is a decomposition of $\mathcal{H}$ with respect to $\mathcal{E}$,

$$\mathcal{H} = \mathcal{H}_{\mathbb{C}} \oplus \mathcal{H}_{\mathbb{C}} j, \tag{2.1}$$

where $\mathcal{H}_{\mathbb{C}} = \overline{\operatorname{span}_{\mathbb{C}}\{\mathcal{E}\}}$ denotes the right $\mathbb{C}$-vector space with orthonormal basis $\mathcal{E}$. Hence, every $u \in \mathcal{H}$ decomposes uniquely as a sum $u = x + yj$, for some $x, y \in \mathcal{H}_{\mathbb{C}}$. For convenience, we sometimes write $\mathcal{H}$ multiplicatively as $\mathcal{H}_{\mathbb{C}}^2$, and represent $u \in \mathcal{H}$ as a pair $(x, y) \in \mathcal{H}_{\mathbb{C}}^2$. Here, $\oplus$ is used as a direct sum of real spaces.

We note that neither $\mathcal{H}_{\mathbb{C}}$ nor $\mathcal{H}_{\mathbb{C}} j$ are $\mathbb{H}$-vector spaces, but $\mathcal{H}_{\mathbb{C}}$ is a complex Hilbert space endowed with the $\mathbb{C}$-inner product given by:

$$\langle x, y\rangle_{\mathbb{C}} := \left\langle \sum_n e_n x_n, \sum_m e_m y_m \right\rangle = \sum_n y_n^* x_n \in \mathbb{C},$$

with $x = \sum_n e_n x_n$ and $y = \sum_n e_n y_n$ in $\mathcal{H}_{\mathbb{C}}$. Clearly, this $\mathbb{C}$-inner product coincides with the inner product of $\mathcal{H}$ when $x, y \in \mathcal{H}_{\mathbb{C}}$.

Definition 2.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a complex operator if there is an orthonormal basis $\mathcal{E} = \{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$ such that

$$\langle T(e_n), e_m\rangle \in \mathbb{C}, \quad \text{for every } n, m \in \mathbb{N}.$$

For convenience, if $\langle Te_n, e_m\rangle \in \mathbb{R}$, for all $n, m \in \mathbb{N}$, we call T a real operator. There are many examples of complex (real) operators, for instance the quaternionic identity operator or the quaternionic backward shift, just to name a couple. For more examples see [CDM6].

Consider both a decomposition of $\mathcal{H}$ as in (2.1) and $T \in \mathcal{B}(\mathcal{H})$ a complex operator with respect to the basis $\mathcal{E}$. We can take the restriction $T_| := T_{|\mathcal{H}_\mathbb{C}} : \mathcal{H}_\mathbb{C} \to \mathcal{H}$ of T, the composition of the maps $T : \mathcal{H} \to \mathcal{H}$ and $\iota : \mathcal{H}_\mathbb{C} \hookrightarrow \mathcal{H}_\mathbb{C} \oplus \mathcal{H}_\mathbb{C} j$, where ι is the injective map $\iota(x) = x + 0j$. Although $T_| = T \circ \iota$ is not a $\mathbb{H}$-linear map, for $\mathcal{H}_\mathbb{C}$ is not a $\mathbb{H}$-linear space, we easily see that it is a right $\mathbb{C}$-linear map on $\mathcal{H}_\mathbb{C}$. Moreover, $T_|(\mathcal{H}_\mathbb{C}) \subseteq \mathcal{H}_\mathbb{C}$. In fact, for $x \in \mathcal{H}_\mathbb{C}$, we have

$$T_|(x) = T(x) = T\left(\sum_n e_n x_n\right) = \sum_n T(e_n)x_n \tag{2.2}$$

and since T is complex,

$$\langle T_|(x), e_m\rangle = \left\langle \sum_n T(e_n)x_n, e_m\right\rangle = \sum_n \langle T(e_n), e_m\rangle x_n \in \mathbb{C},$$

for every $e_m \in \mathcal{E}$. Hence, $T_|(\mathcal{H}_\mathbb{C}) \subseteq \mathcal{H}_\mathbb{C}$ and, for every $x + yj \in \mathcal{H} = \mathcal{H}_\mathbb{C} \oplus \mathcal{H}_\mathbb{C} j$, we have

$$T(x + yj) = T(x) + T(y)j = T_|(x) + T_|(y)j. \tag{2.3}$$

Denote by $\mathcal{B}(\mathcal{H}_\mathbb{C})$ the set of bounded right $\mathbb{C}$-linear operators on $\mathcal{H}_\mathbb{C}$.

Let P_n be the projection of $\mathcal{H}$ onto $\operatorname{span}\{e_1, \dots, e_n\}$. For a complex (real) operator T, let K_n be such that

$$T + K_n = (I - P_n)T(I - P_n), \text{ for each } n \in \mathbb{N}.$$

Since $\langle P_n e_k, e_m\rangle \in \{0, 1\}$, P_n and T are complex (real) operators with respect to the basis $\mathcal{E}$. Hence their composition is also a complex (real) operator with respect to the same basis, and the same holds for the addition of complex (real) operators. Clearly, if we write

$$K_n = -P_nT - TP_n + P_nTP_n \tag{2.4}$$

we see that K_n is a complex (real) compact operator.

Lemma 2.2. *For a real or a complex operator $T \in \mathcal{B}(\mathcal{H})$, the following assertions are true:*

(1) Let $T \in \mathcal{B}(\mathcal{H})$ be a complex operator and K_n the complex compact operator as in (2.4). Let $(\overline{w}_n)_n$ be a sequence with $\overline{w}_n \in \overline{W(T + K_n)}$ (resp. in $\overline{W_\mathbb{C}(T + K_n)}$) and w a sublimit of $\overline{w}_n$. Then $w \in W_e(T)$ (resp. $w \in W_{e,\mathbb{C}}(T)$).

(2) Let $T \in \mathcal{B}(\mathcal{H})$ be a real operator and K_n the real compact operator as in (2.4). Let $(\overline{w}_n)_n$ be a sequence with $\overline{w}_n \in \overline{W_\mathbb{C}(T + K_n)}$ (resp. $\overline{W_\mathbb{R}(T + K_n)}$) and w a sublimit of $\overline{w}_n$. Then $w \in W_{e,\mathbb{C}}(T)$ (resp. $w \in W_{e,\mathbb{R}}(T)$).

Demonstração. We will prove only part (1) in the case when $(\overline{w}_n)_n$ is a sequence in $\overline{W(T + K_n)}$. The remaining cases are proved similarly, following [FSW, Corollary in page 189] for the complex essential numerical range.

If w is a sublimit of $\overline{w}_n$ then there is a subsequence convergent to w, which we still denote by $(\overline{w}_n)_n$ for the sake of simplicity. Take $\varepsilon = \frac{1}{2^p}$ ($p \in \mathbb{N}$). There exist $n_p \in \mathbb{N}$ and

$\overline{w}_{n_p} \in \overline{W(T+K_{n_p})}$ such that

$$|\overline{w}_{n_p} - w| < \frac{1}{2^{p+1}}. \tag{2.5}$$

Note that we may choose $n_p \underset{p}{\to} \infty$. For such n_p, there is $w_{n_p} \in W(T+K_{n_p})$ such that

$$|w_{n_p} - \overline{w}_{n_p}| < \frac{1}{2^{p+1}}. \tag{2.6}$$

Again for simplicity we denote w_{n_p} and K_{n_p} by w_p and K_p.

From (2.5) and (2.6), we have

$$|w_p - w| < \frac{1}{2^p}, \text{ for all } p \in \mathbb{N}.$$

So we have just constructed a sequence $(w_p)_p$ with $w_p \in W(T+K_p)$ and $w_p \to w$.

We will show that $w \in W_e(T)$ by finding an essential sequence for w. Since $w_p \in W(T+K_p)$, there exists $x_p \in \mathcal{S}_{\mathcal{H}}$ and

$$\begin{aligned} w_p &= \langle (T+K_p)x_p, x_p \rangle \\ &= \langle T(I-P_p)x_p, (I-P_p)x_p \rangle \\ &= \langle Ty_p, y_p \rangle, \end{aligned}$$

where $y_p = (I-P_p)x_p$.

We claim that $y_p \rightharpoonup 0$. In fact, given $z \in \mathcal{H}$,

$$\begin{aligned} |\langle y_p, z \rangle| &= |\langle (I-P_p)x_p, z \rangle \\ &= |\langle x_p, (I-P_p)z \rangle| \\ &\leqslant \|(I-P_p)z\|. \end{aligned}$$

Since $\|z\|^2 = \|P_p z\|^2 + \|(I-P_p)z\|^2$ and $\|P_p z\| \to \|z\|$, it follows that $\|(I-P_p)z\| \to 0$. Hence, $\langle y_p, z \rangle \to 0$, for every $z \in \mathcal{H}$.

The proof now splits into two cases.

Case 1: $0 \in W_e(T)$

If $w = 0$ then we are done. Suppose $w \neq 0$. Since

$$\|y_p\| = \|(I-P_p)x_p\| \leqslant 1,$$

$\left(\|y_p\|\right)_p$ has a convergent subsequence, still denoted, for simplicity, by $\left(\|y_p\|\right)_p$. Let $\|y_p\| \to \eta$. If $\eta = 0$ then

$$|w_p| = |\langle Ty_p, y_p \rangle| \leqslant \|Ty_p\| \|y_p\|,$$

and so $|w_p| \to 0$, contradicting our assumption that $w \neq 0$. Hence, $\eta \neq 0$ and we may take $\xi_p = \dfrac{y_p}{\|y_p\|} \in \mathcal{S}_{\mathcal{H}}$. Clearly, $\xi_p \rightharpoonup 0$ because $y_p \rightharpoonup 0$ and $\|y_p\| \to \eta \neq 0$. We have

$$\begin{aligned}\langle T\xi_p, \xi_p\rangle &= \frac{1}{\|y_p\|^2}\langle Ty_p, y_p\rangle \\ &= \frac{1}{\|y_p\|^2} w_p \to \frac{w}{\eta^2}.\end{aligned}$$

We conclude that $(\xi_p)_p$ is an essential sequence for $\dfrac{w}{\eta^2}$ and Theorem 1.1 implies that $\dfrac{w}{\eta^2} \in W_e(T)$. Writing $w = (1-\eta^2)0+\eta^2\frac{w}{\eta^2}$ it follows from the convexity of $W_e(T)$ that $w \in W_e(T)$.

Case 2: $0 \notin W_e(T)$

Since $W_e(T)$ is non empty, we can choose $q, q^* \in W_e(T)$, (see [CDMS1, Proposition 2.5]). Convexity of the essential numerical range (see [CDMS1, Theorem 3.2]) implies that $W_e(T) \cap \mathbb{R} \neq \varnothing$. Then, there exists $\mu \in W_e(T) \cap \mathbb{R}$, that is, $0 \in W_e(T - \mu I)$, see [CDMS1, Proposition 2.5]. But $\overline{w}_p \in \overline{W(T + K_p)}$ and therefore, using again the numerical range translation property, $\overline{w}_p - \mu \in \overline{W(T - \mu I + K_p)}$. We thus fall into **case 1**, replacing T by $T - \mu I$. We conclude that $w - \mu \in W_e(T - \mu I)$, that is, $w \in W_e(T)$.

■

A direct consequence of the previous lemma is that it is enough to intersect over complex compact operators to obtain the quaternionic essential numerical range of a complex operator. Moreover, when T is a real operator, the intersection can be taken over the real compact operators to obtain both the quaternionic and the complex essential numerical ranges. Let $\mathcal{K}_{\mathbb{C}}(\mathcal{H})$ and $\mathcal{K}_{\mathbb{R}}(\mathcal{H})$ denote, respectively, the set of all compact complex and compact real operators in $\mathcal{H}$.

Theorem 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$.*

1. If T is a complex operator, then $W_e(T) = \bigcap_{K \in \mathcal{K}_{\mathbb{C}}(\mathcal{H})} \overline{W(T+K)}$.

2. If T is a real operator, then

(i) $W_e(T) = \bigcap_{K \in \mathcal{K}_{\mathbb{R}}(\mathcal{H})} \overline{W(T+K)}$.

(ii) $W_{e,\mathbb{C}}(T) = \bigcap_{K \in \mathcal{K}_{\mathbb{R}}(\mathcal{H})} \overline{W_{\mathbb{C}}(T+K)}$.

Demonstração. Let T be a complex operator. It is clear that

$$W_e(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \overline{W(T+K)} \subseteq \bigcap_{K \in \mathcal{K}_{\mathbb{C}}(\mathcal{H})} \overline{W(T+K)}.$$

To prove the other inclusion, let $w \in \overline{W(T+K)}$, for all $K \in \mathcal{K}_{\mathbb{C}}(\mathcal{H})$. In particular, $w \in \overline{W(T+K_n)}$, for the complex operators K_n defined in (2.4). The result follows from Lemma 2.2 with $\overline{w}_k = w$, for all $k \in \mathbb{N}$.

A similar argument applies to prove the other statements, using Lemma 2.2.

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Next result characterizes the essential bild of a complex operator $T \in \mathcal{B}(\mathcal{H})$ in terms of the complex numerical range of T and T^*.

Theorem 2.4. *If T is a complex operator, then*

$$B_e(T) = \operatorname{conv}\{W_{e,\mathbb{C}}(T), W_{e,\mathbb{C}}(T^*)\}.$$

Demonstração. We recall that $W_{e,\mathbb{C}}(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H}_\mathbb{C})} \overline{W_\mathbb{C}(T+K)}$, where $\mathcal{K}(\mathcal{H}_\mathbb{C})$ is the set of compact operators on the complex Hilbert space $\mathcal{H}_\mathbb{C}$.

Let us first prove that $\mathcal{K}(\mathcal{H}_\mathbb{C}) \cong \mathcal{K}_\mathbb{C}(\mathcal{H})$. On the one hand, any operator $K \in \mathcal{K}(\mathcal{H}_\mathbb{C})$ is also in $\mathcal{K}_\mathbb{C}(\mathcal{H})$ through an extension $\hat{K} : \mathcal{H} \to \mathcal{H}$ which we next describe. If $K \in \mathcal{K}(\mathcal{H}_\mathbb{C})$, K is a compact operator in the complex Hilbert space $\mathcal{H}_\mathbb{C}$, then $\langle Ke_n, e_m\rangle \in \mathbb{C}$. Considering the vector space $\mathcal{H} = \operatorname{span}_\mathbb{H}\{\mathcal{E}\}$, define the $\mathbb{H}$-linear operator, $\hat{K} : \mathcal{H} \to \mathcal{H}$, by

$$\hat{K}(e_n) := K(e_n).$$

Then clearly $\langle \hat{K}e_n, e_m\rangle = \langle Ke_n, e_m\rangle \in \mathbb{C}$ and thus $\hat{K}$ is a complex and compact operator. On the other hand, any operator $K \in \mathcal{K}_\mathbb{C}(\mathcal{H})$ is also in $\mathcal{K}(\mathcal{H}_\mathbb{C})$ through its restriction $K_| : \mathcal{H}_\mathbb{C} \to \mathcal{H}_\mathbb{C}$ described in (2.2). Hence $\mathcal{K}_\mathbb{C}(\mathcal{H}) \cong \mathcal{K}(\mathcal{H}_\mathbb{C})$. In fact, the processes of extension and restriction are two isometries, with each serving as the inverse of the other. Consequently, these two spaces are isometrically isomorphic.

Furthermore, from the fact that $K(x) = \hat{K}(x)$ for $x \in \mathcal{H}_\mathbb{C}$, we have

$$\langle (T+K)\,x, x\rangle = \langle (T+\hat{K})x, x\rangle,$$

we conclude that

$$W_{e,\mathbb{C}}(T) = \bigcap_{K \in \mathcal{K}_\mathbb{C}(\mathcal{H})} \overline{W_\mathbb{C}(T+K)}.$$

It is common knowledge that $W_\mathbb{C}(T+K) \subseteq W(T+K) \cap \mathbb{C}$, and from Theorem 2.3 we have $W_{e,\mathbb{C}}(T) \subseteq B_e(T)$. Since $W_e(T^*) = W_e(T)$, see [CDMS1, Proposition 2.5 (iii)], then also $W_{e,\mathbb{C}}(T^*) \subseteq B_e(T)$. Using that the essential bild $B_e(T)$ is convex, we conclude that

$$\operatorname{conv}\{W_{e,\mathbb{C}}(T), W_{e,\mathbb{C}}(T^*)\} \subseteq B_e(T).$$

Let us now prove the converse inclusion. From Theorem 2.3 and [CDM6, page 11] we have

$$\begin{aligned} B_e(T) &= \bigcap_{K \in \mathcal{K}_\mathbb{C}(\mathcal{H})} \overline{B(T+K)} \\ &\subseteq \bigcap_{K \in \mathcal{K}_\mathbb{C}(\mathcal{H})} \overline{\operatorname{conv}\{W_\mathbb{C}(T+K), W_\mathbb{C}(T^*+K^*)\}}. \end{aligned}$$

Then,

$$B_e(T) \subseteq \bigcap_{K \in \mathcal{K}_\mathbb{C}(\mathcal{H})} \operatorname{conv}\{\overline{W_\mathbb{C}(T+K)}, \overline{W_\mathbb{C}(T^*+K^*)}\}.$$

Take $w \in B_e(T)$. Then w is a convex combination of elements in $\overline{W_{\mathbb{C}}(T+K)}$ and $\overline{W_{\mathbb{C}}(T^*+K^*)}$ for every complex compact operator K. In particular, for K_n as in (2.4), we may write

$$\omega = \alpha_n^1\omega_n^1 + \alpha_n^2\omega_n^2, \tag{2.7}$$

where $\alpha_n^1, \alpha_n^2 \geqslant 0$, $\alpha_n^1 + \alpha_n^2 = 1$, $\omega_n^1 \in \overline{W_{\mathbb{C}}(T+K_n)}$ and $\omega_n^2 \in \overline{W_{\mathbb{C}}(T^*+K_n^*)}$.

Since $\omega_n^1, w_n^2 \in \overline{B(0,\|T\|)}$, we may take subsequences, denoted by $(w_n^1)_n$ and $(w_n^2)_n$ for the sake of simplicity, convergent to, say, ω^1 and ω^2, respectively. From Lemma 2.2 we have $\omega^1 \in W_{e,\mathbb{C}}(T)$ and $\omega^2 \in W_{e,\mathbb{C}}(T^*)$. Moreover, from the fact that $\alpha_n^i \in [0,1]$, $i = 1,2$, there are also convergent subsequences, still denoted by α_n^i, such that $\alpha_n^i \to \alpha^i$, $i = 1,2$. Taking limits in (2.7) we conclude that $\omega = \alpha^1\omega^1 + \alpha^2\omega^2$, and the result follows.

■

The previous result implies that the essential bild of a real operator T coincides with its complex essential numerical range. Furthermore, since $B_e(T)$ intersects the real line, we have $W_{e,\mathbb{C}}(T) \cap \mathbb{R} \neq \varnothing$.

Corollary 2.5. *If T is a real operator, then $B_e(T) = W_{e,\mathbb{C}}(T)$.*

As an example, let us apply the above result to compute the essential bild of the unilateral quaternionic shift.

Example 2.6. The unilateral scalar weighted (quaternionic) shift on $\mathcal{H}$ with respect to an orthonormal basis $\{e_n\}_n$ is defined by

$$Se_n = e_{n+1}w_n, w_n \in \mathbb{H}, n \geqslant 1.$$

We denote $S \sim_u \{w_k\}_{k=1}^\infty$ for an operator S unitarily equivalent to the weighted shift whose sequence of weights is $\{w_k\}_{k=1}^\infty$.

Taking the unitary operator $U = \operatorname{diag}(1, c_1, c_1c_2, \ldots)$, where $c_n = \dfrac{w_n^*}{|w_n|}$ $(n \geqslant 1)$, one easily checks that $USU^*e_n = |w_n|e_{n+1}$, i.e, $S \sim_u \{|w_k|\}_{k=1}^\infty$. Thus, we may assume that S is real.

Let us consider two special cases. First, we suppose that the sequence $\{|w_k|\}_{k=1}^\infty$ converges to $a \geqslant 0$. By part (a) of [WW, Proposition 2.2] we know that $W_{e,\mathbb{C}}(S) = \overline{\mathbb{D}(0,a)}$. Applying Corollary 2.5, we have that $B_e(S) = \overline{\mathbb{D}(0,a)}$.

For the second case assume that $\{w_k\}_{k=1}^\infty$ is a periodic injective shift, that is, $w_k \neq 0$, for every k. It is known that $W_{\mathbb{C}}(S)$ is an open circular disk about the origin, see [S, Proposition 6] and [R]. Therefore, from [S, Proposition 7] and Corollary 2.5, it follows that $B_e(S)$ is a closed circular disk about the origin.

We can also find the quaternionic essential numerical range of a complex block diagonal operator using the results from [CDMS2].

Example 2.7. Let $A \in \mathcal{M}_k(\mathbb{C})$ and let $T = \bigoplus_{n\in\mathbb{N}} A$ be a complex block diagonal operator. From [CDMS2, Example 4], we know that $W_{e,\mathbb{C}}(T) = W(A)$. Then, using Theorem 2.4, we conclude that

$$B_e(T) = \operatorname{conv}\{W_{\mathbb{C}}(A), W_{\mathbb{C}}(A^*)\}.$$

Theorem 2 in [CDMS2] can be extended to the quaternionic setting. We omit the proof, since it follows along the same lines.

Theorem 2.8. *Let $T = \bigoplus_{n\in\mathbb{N}} A_n \in \mathcal{B}(\mathcal{H})$. Then,*

$$W_e(T) = \operatorname{conv}\Big(\bigcap_{k\geqslant 1}\overline{\bigcup_{n\geqslant k} W(A_n)}\Big).$$

A simple application of the above theorem allows us to observe that the quaternionic essential numerical range of a quaternionic block diagonal operator $T = \bigoplus_{n\in\mathbb{N}} A$, where $A \in \mathcal{M}_k(\mathbb{H})$, is given by $W_e(T) = \operatorname{conv}(W(A))$. In particular, in the space of quaternionic sequences $\mathcal{H} = \ell^2(\mathbb{H})$, if T is the operator of pointwise left multiplication by $q \in \mathbb{H}$, that is, $T = qI$, where I denotes the identity operator, the quaternionic essential numerical range can be easily computed.

Example 2.9. Let $T = qI$, with $q \in \mathbb{H}$, so that T is the block diagonal operator $T = \bigoplus_{n\in\mathbb{N}} A_n$, where $A_n = [q] \in \mathcal{M}_1(\mathbb{H})$, for all $n \in \mathbb{N}$. We can write $q = q_0 + q_{\mathrm{Im}} \in \mathbb{H}$, where $q_0 = \operatorname{Re}(q)$ and $q_{\mathrm{Im}} = \operatorname{Im}(q)$. From Theorem 2.8, we have that

$$W_e(qI) = \operatorname{conv}\Big(\overline{W(q)}\Big) = \overline{\mathbb{D}(q_0, |q_{\mathrm{Im}}|)},$$

where we used the well known fact that $W(q) = \partial\mathbb{D}(q_0, |q_{\mathrm{Im}}|)$.

We can also conclude that $\overline{\mathbb{D}(q_0, |q_{\mathrm{Im}}|)} = W_e(qI) \subseteq \overline{W(qI)} \subseteq \overline{\mathbb{D}(q_0, |q_{\mathrm{Im}}|)}$, that is, $\overline{W(qI)} = \overline{\mathbb{D}(q_0, |q_{\mathrm{Im}}|)}$.

3. Essential S-spectrum and essential numerical range

The right multiplication by a complex number λ is the right $\mathbb{C}$-linear operator $I_{\mathcal{H}_\mathbb{C}} \cdot \lambda \in \mathcal{B}(\mathcal{H}_\mathbb{C})$ given by $(I_{\mathcal{H}_\mathbb{C}} \cdot \lambda)(x) = x\lambda$, for every $x \in \mathcal{H}_\mathbb{C}$, where $I_{\mathcal{H}_\mathbb{C}}$ denotes the identity operator on $\mathcal{H}_\mathbb{C}$. In particular, if $T \in \mathcal{B}(\mathcal{H}_\mathbb{C})$ is a complex operator, then for every $\lambda \in \mathbb{C}$, $T - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda$ belongs to $\mathcal{B}(\mathcal{H}_\mathbb{C})$. Next definition of complex essential spectrum is the natural notion of spectrum in right complex Hilbert spaces.

Definition 3.1. For any complex operator $T \in \mathcal{B}(\mathcal{H})$, the complex essential spectrum of T is the subset of $\mathbb{C}$ given by

$$\sigma_e(T) := \sigma_e(T_|) = \{\lambda \in \mathbb{C} : T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda \text{ is not Fredholm in } \mathcal{B}(\mathcal{H}_\mathbb{C})\}.$$

Recall that in (1.1) we defined the essential S-spectrum of $T \in \mathcal{B}(\mathcal{H})$ by

$$\sigma_e^S(T) = \{q \in \mathbb{H} : \Delta_q(T) \notin \mathcal{F}(\mathcal{H})\}.$$

To characterize the essential S-spectrum of a complex operator in terms of the complex essential spectrum, observe that $\Delta_\lambda(T)$ is also a complex operator and hence we can consider its restriction to $\mathcal{H}_\mathbb{C}$,

$$\Delta_\lambda(T)_| : \mathcal{H}_\mathbb{C} \to \mathcal{H}_\mathbb{C}.$$

We have

$$\Delta_\lambda(T)(\mathcal{H}_\mathbb{C}) = \Delta_\lambda(T)_|(\mathcal{H}_\mathbb{C}) = \Delta_\lambda(T_|)(\mathcal{H}_\mathbb{C}) \tag{3.1}$$

and so, it follows from the decomposition $\mathcal{H} = \mathcal{H}_\mathbb{C} \oplus \mathcal{H}_\mathbb{C} j$ that

$$\begin{aligned} \Delta_\lambda(T)(\mathcal{H}) &= \Delta_\lambda(T)(\mathcal{H}_\mathbb{C}) \oplus \Delta_\lambda(T)\mathcal{H}_\mathbb{C} j \\ &= \Delta_\lambda(T_|)(\mathcal{H}_\mathbb{C}) \oplus \Delta_\lambda(T_|)\mathcal{H}_\mathbb{C} j. \end{aligned}$$

Proposition 3.2. *Let T be a complex operator. Then*

$$\sigma_e^S(T) = [\sigma_e(T)],$$

where $[\sigma_e(T)] = \bigcup_{\lambda\in\sigma_e(T)}[\lambda]$.

Demonstração. Let us start by proving the inclusion $\sigma_e^S(T) \subseteq [\sigma_e(T)]$. Since $\sigma_e^S(T)$ is circular, it is enough to take $\lambda \in \sigma_e^S(T) \cap \mathbb{C}$ and prove that $\lambda \in \sigma_e(T)$, i.e. $T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda$ is not Fredholm in $\mathcal{B}(\mathcal{H}_\mathbb{C})$.

We will first show that the operator $\Delta_\lambda(T)_|$ is not Fredholm in $\mathcal{B}(\mathcal{H}_\mathbb{C})$. In fact, if $\Delta_\lambda(T)_|$ were Fredholm in $\mathcal{B}(\mathcal{H}_\mathbb{C})$ then $\dim\ker(\Delta_\lambda(T)_|) < \infty$ and $\dim\operatorname{coker}(\Delta_\lambda(T)_|) < \infty$.

In case $\dim\ker(\Delta_\lambda(T)_|)$ were finite then, using that

$$\ker(\Delta_\lambda(T)) = \ker(\Delta_\lambda(T)_|) \oplus \ker(\Delta_\lambda(T)_|)j,$$

it would follow that $\dim\ker(\Delta_\lambda(T))$ would also be finite. A similar argument shows that in case $\dim\operatorname{coker}(\Delta_\lambda(T)_|) < \infty$ then $\dim\operatorname{coker}(\Delta_\lambda(T)) < \infty$ using that

$$\operatorname{coker}(\Delta_\lambda(T)) = \operatorname{coker}(\Delta_\lambda(T)_|) \oplus \operatorname{coker}(\Delta_\lambda(T)_|)j.$$

So in either case $\Delta_\lambda(T)$ would be Fredholm in $\mathcal{B}(\mathcal{H})$, leading us to a contradiction.

Now we are able to prove that $T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda$ is not Fredholm in $\mathcal{B}(\mathcal{H}_\mathbb{C})$. Since T is a complex operator, we can write

$$\Delta_\lambda(T_|) = (T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda^*)(T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda). \tag{3.2}$$

Therefore, $\pi(T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda^*)\pi(T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda)$ is not invertible in $\mathcal{B}(\mathcal{H}_\mathbb{C})/\mathcal{K}(\mathcal{H}_\mathbb{C})$, that is, either $\pi(T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda^*)$ or $\pi(T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda)$ is not invertible in $\mathcal{B}(\mathcal{H}_\mathbb{C})/\mathcal{K}(\mathcal{H}_\mathbb{C})$. It follows that either $T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda$ or $T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda^*$ is not Fredholm in $\mathcal{B}(\mathcal{H}_\mathbb{C})$, i.e. λ or λ^* is in $\sigma_e(T)$. Hence, $\lambda \in [\sigma_e(T)]$.

To prove the converse inclusion suppose that $\lambda \notin \sigma_e^S(T)$, that is $\Delta_\lambda(T)$ is Fredholm in $\mathcal{B}(\mathcal{H})$. Since $\dim\ker(\Delta_\lambda(T)) < \infty$ then $\ker(\Delta_\lambda(T_|))$ is a finite $\mathbb{H}-$ subspace, for $\ker(\Delta_\lambda(T_|)) \subset \ker(\Delta_\lambda(T))$ as $\mathbb{H}-$ subspaces. Clearly, $\ker(\Delta_\lambda(T_|))$ is also a finite $\mathbb{C}-$ subspace. Because

$\Delta_\lambda(T_|) \in \mathcal{B}(\mathcal{H}_\mathbb{C})$, we can use (3.2) to conclude that $\ker(T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda) \subseteq \ker(\Delta_\lambda(T_|))$. It follows that $\dim\ker(T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda)$ is finite.

The same reasoning proves that $\dim \operatorname{coker}(\Delta_\lambda(T)) < \infty$ then $\dim \operatorname{coker}(T_|^* - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda) < \infty$. In this case, we use the fact that $\Delta_\lambda(T)$ can be written in the form $\Delta_\lambda(T_|) = (T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda)(T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda^*)$.

Both $\dim\ker(T_| - I_{\mathcal{H}_\mathbb{C}} \cdot \lambda)$ and $\dim \operatorname{coker}(\Delta_\lambda(T)) < \infty$ allows us to conclude that $\lambda \notin \sigma_e(T)$. ■

As we have seen in Example 2.6, known results in the complex setting regarding the numerical range can be used to reach similar ones in the quaternionic setting. The same can be said about the S-spectrum.

Example 3.3. Let S be an injective unilateral quaternionic scalar weighted shift. Then, as seen in Example 2.6, $S \sim_u \{|w_k|\}_{k=1}^\infty$. From the above proposition the essential S-spectrum of S is the set of the similarity classes of $\sigma_e(S)$, known to be the annulus (see [LJS, p. 417])

$$\{\lambda \in \mathbb{C} : r_1(S) \leqslant |\lambda| \leqslant r(S)\},$$

where $r(S)$ is the spectral radius, $r_1(S) = \lim_k (m(S^k))^{1/k}$ and $m(S) = \inf\{\|Sx\| : \|x\| = 1\}$ is called the lower bound of S.

In [CDMS1, Theorem 2.6] the authors prove that $\sigma_e^S(T) \subseteq W_e(T)$, for every $T \in \mathcal{B}(\mathcal{H})$. The convexity of the essential numerical range [CDMS1, Theorem 3.2] implies that $\operatorname{conv}\{\sigma_e^S(T)\} \subseteq W_e(T)$. When T is a quaternionic normal operator this inclusion is an equality, as we next prove. We observe that this class of operators is contained in the class of complex operators, up to approximate unitary equivalence. Thus, before we need to show that the essential S-spectrum and the essential numerical range are invariant under this equivalence relation. Recall that two operators $T, R \in \mathcal{B}(\mathcal{H})$ are approximate unitary equivalent, and we write $T \sim_a R$, if there exists a sequence of unitary operators $(U_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|U_n R U_n^* - T\| = 0$.

Proposition 3.4. *If $T, R \in \mathcal{B}(\mathcal{H})$ are such that $T \sim_a R$ then*

(i) $\sigma_e^S(R) = \sigma_e^S(T)$,
(ii) $W_e(R) = W_e(T)$.

Demonstração.

Let $(U_n)_n$ be a sequence of unitary operators such that $R_n = U_n R U_n^* \to T$.

Let us first prove (i). We will start by showing that $\limsup \sigma_e^S(R_n) \subseteq \sigma_e^S(T)$. The equality then follows from symmetry. Recall furthermore that $\limsup \sigma_e^S(R_n)$ is a closed subset of $\mathbb{H}$ and that $q \in \limsup \sigma_e^S(R_n)$ if, and only if, there is an increasing sequence $(n_k)_k$ of positive integers such that $q_{n_k} \in \sigma_e^S(R_{n_k})$ for all k and $q_{n_k} \to q$.

Let $q \in \limsup \sigma_e^S(R_n)$ and suppose $q \notin \sigma_e^S(T)$. The set $\mathcal{F}(\mathcal{H})$ of Fredholm operators is the inverse image, under the canonical surjection $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, of the open set formed by the group of invertible elements in the Calkin algebra. Since π is continuous then $\mathcal{F}(\mathcal{H})$ is open. Hence, there is $\varepsilon > 0$ such that $B_\varepsilon(\Delta_q(T)) \subset \mathcal{F}(\mathcal{H})$.

Now, let $(n_k)_k$ be an increasing sequence of positive integers such that $q_{n_k} \in \sigma_e^S(R_{n_k})$ for all k and $q_{n_k} \to q$. Since $R_n \to T$, there is $k_0 \in \mathbb{N}$ such that $\Delta_{q_{n_k}}(R_{n_k}) \in B_\varepsilon(\Delta_q(T))$ for all $k \geqslant k_0$. In fact, fully writing $\Delta_{q_{n_k}}(R_{n_k})$ and $\Delta_q(T)$, we see that $\|\Delta_{q_{n_k}}(R_{n_k}) - \Delta_q(T)\| \to 0$ as $k \to \infty$. So, for $k \geqslant k_0$, $q_{n_k} \notin \sigma_e^S(R_{n_k})$ and we get a contradiction.

Then, $\sigma_e^S(R) = \limsup \sigma_e^S(R_n) \subseteq \sigma_e^S(T)$.

We now prove (ii). Let $q \in W_e(T)$. By Theorem 1.1 (iii), there exists an orthonormal sequence $(e_n)_n$ such that $q = \lim_{n\to\infty}\langle Te_n, e_n\rangle$. For a fixed $m \in \mathbb{N}$, the sequence $\Big(\langle U_m^* R U_m e_n, e_n\rangle\Big)_{n\geqslant 1}$ is bounded and hence it has a subsequence $\Big(\langle U_m^* R U_m e_{n_k}, e_{n_k}\rangle\Big)_{k\geqslant 1}$ convergent to, say, μ_m. We have

$$\begin{aligned} |\mu_m - q| &= |\lim_{k\to\infty}\langle (U_m^* R U_m - T)e_{n_k}, e_{n_k}\rangle| \\ &\leqslant \|U_m^* R U_m - T\|. \end{aligned}$$

Since $T \sim_a R$ it follows that $\lim_{m\to\infty}\mu_m = q$. By [CDMS1, Lemma 2.1], we have $\mu_m = \lim_{k\to\infty}\langle (U_m^* R U_m + K)e_{n_k}, e_{n_k}\rangle$, for every compact operator $K \in \mathcal{K}(\mathcal{H})$. Therefore, $\mu_m \in \overline{W(U_m^* R U_m + K)}$, for every compact operator $K \in \mathcal{K}(\mathcal{H})$ and so $\mu_m \in W_e(U_m^* R U_m)$. Since $W_e(U_m^* R U_m) = W_e(R)$ and this is a closed set, we must have $q \in W_e(R)$.

We have proved that $W_e(T) \subseteq W_e(R)$ and the converse inclusion follows from symmetry. ■

Theorem 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then*

$$W_e(T) = \operatorname{conv}\{\sigma_e^S(T)\}.$$

Demonstração. As observed above, it remains to prove that

$$W_e(T) \subseteq \operatorname{conv}\{\sigma_e^S(T)\}.$$

From Proposition 4.1 in [CDM6] there exists a diagonal operator $D \in \mathcal{B}(\mathcal{H})$ with respect to an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$, where $D(e_n) = e_n d_n$, with $d_n \in \mathbb{C}^+$, such that $T \sim_a D$. By Proposition 3.4, we have $W_e(T) = W_e(D)$ and $\sigma_e^S(T) = \sigma_e^S(D)$. Hence it is enough to show that $W_e(D) \subseteq \operatorname{conv}\{\sigma_e^S(D)\}$, that is,

$$B_e(D) \subseteq \operatorname{conv}\{\sigma_e^S(D)\} \cap \mathbb{C}. \tag{3.3}$$

From Theorem 2.4, we have

$$B_e(D) = \operatorname{conv}\{W_{e,\mathbb{C}}(D), W_{e,\mathbb{C}}(D^*)\}.$$

Using [SW, Theorem 8], we have $W_{e,\mathbb{C}}(D) = \operatorname{conv}\{\sigma_e(D)\} \subset \mathbb{C}^+$ and $W_{e,\mathbb{C}}(D^*) = \operatorname{conv}\{\sigma_e(D^*)\} \subset \mathbb{C}^-$. Therefore, $B_e(D) = \operatorname{conv}\{\sigma_e(D), \sigma_e(D^*)\}$. Proposition 3.2 and the fact that $\sigma_e^S(D) =$

$\sigma_e^S(D^*)$ (see [MT]), imply that $\sigma_e(D) = \sigma_e^S(D) \cap \mathbb{C}^+$ and $\sigma_e(D^*) = \sigma_e^S(D) \cap \mathbb{C}^-$. Then

$$\begin{aligned} B_e(D) &= \operatorname{conv}\{\sigma_e^S(D) \cap \mathbb{C}^+, \sigma_e^S(D) \cap \mathbb{C}^-\}, \\ &= \operatorname{conv}\{\sigma_e^S(D)\} \cap \mathbb{C}. \end{aligned}$$

■

Competing interests declaration

The authors report there are no competing interests to declare.

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