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Comparison of Black-Scholes and Heston Models

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Master in Finance

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Setembro, 2023

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Abstract

The goal of this dissertation is to compare the Black-Scholes and the Heston model using Deterministic Volatility Functions (DVF) on option pricing. It is important to emphasize that consistency in the choice of loss functions is crucial. On one hand, for any given model, the loss function should be the same for the parameter estimation and model evaluation, otherwise suboptimal parameter estimates can happen. On the other hand, the estimation of loss functions should be identical across models, in order to avoid inappropriate comparisons. Therefore, it will be used three different loss functions in order to estimate and evaluate which of these option valuation models is the most accurate. The sample data contains S&P 500 Index options traded on Chicago Board Options Exchange (CBOE) and it was considered some exclusionary criteria as suggested by Dumas et al. (1998). The remaining data needed to price options was the risk-free rate for each option maturity and the S&P 500 estimated dividend-yield. For both models, the practical application starts with the usage of the Ordinary Least Squares (OLS), with the objective to minimize the Implied Volatility Root Mean Squared Error for each DVF. Secondly, the objective was to minimize the Dollar Root Mean Squared Error and the Percentage Root Mean Squared Error using the Non-linear Least Squares (NLS), for each DVF. For the Heston model, the parameters are estimated using the loss functions, to get the quoted option prices as close to the model option values as possible. After estimating the loss functions, the objective is to decide which model is the most accurate for option pricing.

Keywords: Black-Scholes; Heston; Loss Functions; Volatility

Resumo

O objetivo desta dissertação foca-se em comparar os modelos Black-Scholes e Heston através do uso de Funções Determinísticas de Volatilidade (DVF) na avaliação de opções financeiras. É importante enfatizar que a consistência na escolha das *loss functions* é crucial. Por um lado, a *loss function* deve ser a mesma para a estimação dos parâmetros e avaliação do modelo, caso contrário, podem acontecer estimativas de parâmetros abaixo do nível ideal. Por outro lado, a estimação das *loss functions* deve ser idêntica entre os modelos, de modo a evitar comparações inadequadas. Deste modo, serão utilizadas três diferentes *loss functions* para estimar e avaliar qual destes modelos de avaliação de opções é o mais preciso. Os dados da amostra contêm opções financeiras do índice S&P 500 negociadas na *Chicago Board Options Exchange* (CBOE) e foram considerados alguns critérios de exclusão sugeridos por Dumas et al. (1998). Os restantes dados necessários para proceder à avaliação das opções financeiras foram a taxa de risco para cada nível de maturidade da opção e o rendimento estimado de dividendos do S&P 500. Para ambos os modelos, a aplicação prática inicia-se com a utilização dos *Ordinary Least Squares (OLS),* com o objetivo de minimizar o erro médio quadrático da raiz da volatilidade implícita para cada DVF. Em segundo lugar, o objetivo foi minimizar o erro médio quadrático da raiz do dólar e o erro médio quadrático percentual usando o *Non-linear Least Squares (NLS)*, para cada DVF. Para o modelo de Heston, os parâmetros são estimados utilizando as *loss functions*, para obter os preços das opções cotadas o mais aproximado possível dos valores das opções avaliadas no modelo. Após estimar as *loss functions*, o objetivo passa por decidir qual o modelo mais preciso para avaliação de opções financeiras.

Palavras-chave: Black-Scholes; Heston; Loss Functions; Volatility

Acronyms

$MSE – Dollar Mean Squared Error

%MSE – Percentage Mean Squared Error

AIC - Akaike Information Criterion

ATM – At the Money

BFM – Best Fitted Model

BSM – Black-Scholes Model

CBOE - Chicago Board Options Exchange

DTM – Days to Maturity

DVF - Deterministic Volatility Functions

EMM - Efficient Method of Moments

FED – Federal Reserve System

GMM - Generalized Method of Moments

ITM – In the Money

IV – Implied Volatility

IVMSE – Implied Volatility Mean Squared Error

NLS - Non-linear Least Squares

NSS - Nelson-Siegel-Svensson

OLS - Ordinary Least Squares

OTM – Out of the Money

PBS - Practitioner Black-Scholes

RMSE - Root Mean Squared Error

RMSVE - Root Mean Squared Valuation Error

TAM - Time Adjusted Moneyness

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Introduction

The economy is no exception to the fact that we now live in a global civilization that affects practically all facets of life. Since the world's largest economies depend on one another and frequently cooperate, there are actual global markets where the prices of financial products are always fluctuating, with financial options being no exception. The market indices record changes that are reported on option exchanges and serve as a reliable signal for market analysis. In practice, the returns reveal skewness and kurtosis, which the model does not consider. When the implied volatility from the Black and Scholes (1973) equation is displayed with respect to the time to maturity and strike price of options, the volatility surface is flat. Therefore, as it is known, in the last four decades, option pricing has witnessed an explosion of new models that address the empirical shortcomings of the restrictive Black-Scholes model.

There are numerous choices available for each of the assumptions of the Black-Scholes model. For instance, the underlying pricing may proceed according to a discrete-time or continuous-time process. It can be a Poisson jump process or a diffusion process, a Markov jump process or a non-Markov jump process, or a combination of jump and diffusion components with or without stochastic volatility and with or without random leaps. Similar numbers of options exist for interest rate term structure.

In fact, the Black-Scholes model is frequently used for option pricing. The foundation of this model, however, is based on a variety of assumptions that are not applicable to actual financial markets. It considers that until the options reach maturity, the volatility remains constant. Because of the volatility skew of stock options, the Black-Scholes model usually overvalues out-the-money (OTM) and undervalues in-the-money (ITM) options if the VIX value is used.

Therefore, one strategy was to make volatility a stochastic quantity by introducing uncertainty into its behavior. A stochastic process can be used to estimate prices that are near to market values by calculating its parameters. Heston (1993) proposed one of the most used stochastic volatility models for pricing options. The major characteristic that distinguishes stochastic volatility models from other models is the assumption that volatility is random rather than constant. For example, the SABR, Chen, and GARCH models are more varieties of stochastic volatility models.

Although the search for the ideal option pricing model can seem never-ending, the present dissertation was carried out to conduct a comprehensive empirical study on the relative merits of testing some different specifications of the Deterministic Volatility Functions (DVF) for the Black-Scholes model and the Heston model, in order to estimate which is the most accurate for option pricing.

For that matter, the structure, and guidelines of the paper “Implied Volatility Functions: Empirical Tests”, from Dumas et al. (1998) will be considered. On the similar manner, the format, and principles of *Fabrice Douglas Rouah's* book (2013), "The Heston Model and Its Extensions in MATLAB and C#," will also be taken into consideration. In that way, it will be developed some codes/scripts in *MATLAB*, in order to test whether the conclusion from the collected data matches the insights for the Black-Scholes model and Heston model, taken by the authors from the papers referred above, and compare them to determine the most accurate model.

Since many authors have the opinion that the Black-Scholes valuation formula no longer holds in the financial markets, due to the constant variance assumption, it will be estimated the implied volatility functions from the DVF option valuation model, to compare with the observed prices from collected data. According to the authors, having an option valuation framework with nonconstant volatility would only be possible when the “volatility of the underlying asset’s return is a deterministic function of the asset price and/or time to expiration”. That said, it will be assessed the stability of the implied volatility functions.

Regarding the Heston model, the parameters are estimated, to get the quoted option prices as close to the model option values as possible. Prices can also be replaced with quoted, and model estimated volatilities. This method uses the error between quoted market prices and model prices, or between market and model implied volatilities. In order to respect the constraints on the parameters, a constrained minimization algorithm must be used.

In this dissertation, it will also be given consideration to the paper “The importance of the loss function in option valuation”, from Christoffersen and Jacobs (2004). For that matter, an analysis will also be carried out to test and compute the Root Mean Squared Error (RMSE) from in-sample and out-of-sample data for 1, 5 and 20 days, taken different loss functions into consideration.

The sample data contains S&P 500 Index options traded on Chicago Board Options Exchange (CBOE) and it was considered some exclusionary criteria as Dumas et al. (1998) suggest. The remaining data needed to price options was the risk-free rate for each option maturity and the S&P 500 estimated dividend-yield.

Literature Review

Despite how helpful past investigations have been, they are always looking backwards and projecting future behavior. Using reported option prices to infer volatility expectations is an alternate strategy, but one that has received less attention in the literature. Because predicted future volatility depends critically on expected future volatility, market participants' expectations of volatility can be recovered by inverting the option valuation formula.

Louis Bachelier introduced the Brownian motion to the financial markets in 1900. Fischer Black and Myron Scholes proposed a well-known model in 1973 for pricing European options based on Geometric Brownian Motion. The logarithm of a quantity that is randomly fluctuating, such as the stock price, moves in a geometric Brownian motion. It makes the supposition that the stock market's volatility will remain constant, and that the distribution of logarithmic returns will be uniform. Therefore, the volatility expectation derived from reported option prices depends on the assumptions underlying the option valuation formula.

The Black-Scholes model presupposes that the price of the asset moves in a geometric Brownian motion with constant volatility. As a result, the implied volatility is the same on all options on the same asset. Black-Scholes implied volatilities, however, tend to differ across exercise prices and times to expiration. However, the implied volatility surface is really skewed, meaning that the volatility varies depending on the strike price and time till maturity. Numerous attempts were made to develop models that would better estimate the option pricing as a result of this mismatch.

For instance, S&P 500 option-implied volatilities display a "smile" pattern before the market crash in October 1987. Higher implied volatilities occur to options that are deeply ITM or OTM rather than to options that are at the money (ATM). The implied volatilities decline monotonically as the exercise price increases in relation to the index level after the crash, with the rate of decrease accelerating for options with shorter time to expiration.

It is believed that the Black-Scholes model's assumption of constant volatility is what prevents it from accurately describing the structure of reported option prices. Volatility is known to decrease when stock values rise, and vice versa. However, it is not simple to account for nonconstant volatility within an option valuation framework. With stochastic volatility, option valuation typically calls for a market price of the risk parameter, which is challenging to measure among other things. An exception occurs when volatility is a deterministic function of asset price and/or time.

Option valuation using the Black-Scholes partial differential equation is still feasible in this situation, though not using the Black-Scholes formula itself. This unique situation is known as the "deterministic volatility function" hypothesis. Derman and Kani (1994a), Dupire (1994), and Rubinstein (1994) develop variations of the DVF approach. Their techniques try to interpret the cross section of option prices and determine the volatility's predicted future behavior.

Instead of proposing a structural form for the volatility function, they look for a lattice of binomial or trinomial equations that precisely fits reported option prices across the cross-section. For example, Rubinstein (1994) makes use of an "implied binomial tree" whose branches at each node are intended to reflect the time change of volatility (either through the selection of up-and-down increment sizes or probabilities). Assessing the time-series validity of the hypothesis that volatility is a deterministic function of asset price and time was the aim of Dumas et al. (1998). This strategy is a potent statistical method that more quickly reaches a conclusion regarding the applicability of the DVF strategy. Dumas et al. (1998) use this strategy to determine whether the volatility function implied today is the same as that embedded in option pricing tomorrow by moving out-of-sample.

If the estimated volatility function is stable over time, this conclusion supports the DVF approach as a crucial new method for determining the mechanism that drives financial market prices as well as for determining hedge ratios and exotic option pricing. On the other hand, if the estimated function is not stable, it can be inferred that the DVF approach to risk management and valuation is incorrect, and it is necessary to look for alternative reasons for the patterns of Black-Scholes implied volatility.

Deterministic volatility functions should be considered when valuing options due to the Black-Scholes model's apparent shortcomings. These deficiencies are most commonly expressed in cross section as the relation between the Black-Scholes implied volatility and option exercise price. Higher maturities typically have lower implied volatilities than options with smaller maturities. This pattern implies that the DVF model's local volatility rate is a function of time. It may be economically relevant for implied volatilities to vary across exercise prices.

For instance, the short-term ITM call's bid-implied volatility typically exceeds the short-term ATM call's ask-implied volatility, indicating a potential for profit through arbitrage. However, selling ITM calls and purchasing ATM calls is a more complex strategy that involves dynamic rebalancing over time to take advantage of the "arbitrage opportunity" than just spreading the options. However, as can be seen in Constantinides (1997), the costs of dynamic rebalancing cannot explain the disparities in implied volatilities since they are too large.

In addition to the DVF approach considered in Dumas et al. (1998) study, several option valuation models can explain the previously observed behavior documented. When the asset price and volatility are negatively correlated, they can be explained by stochastic volatility models like those of Hull and White (1987) and Heston (1993), for instance. The Heston (1993) model contains two stochastic processes, one for the stock price and the other for volatility, in contrast to Black-Scholes' model, which only had the stock price following a stochastic process. Jumps in the stock price dynamics were included to the model by Bates (1996a), which is helpful for pricing out-of-the-money options. However, when the model's complexity rises, more parameters must be estimated, some of which may not be applicable in practice.

Indeed, similar patterns can be produced by the Bates (1996b) jump model when the mean jump is negative. However, deterministic volatility models are the most straightforward since they maintain the arbitrage theory that forms the basis of the Black-Scholes model. They do not call for extra presumptions on investor risk preferences or additional assets that can be used to hedge volatility or jump risk, in contrast to stochastic volatility and jump models.

Given that option valuation models can be used for a variety of purposes, including hedging, speculation, or market manipulation, the choice of loss function is particularly crucial when estimating these models. Naturally, different goals imply various loss functions. One could anticipate that the choice of loss function would be a strongly contested topic in the option valuation literature because the specification of a loss function implicitly equates to the specification of a statistical model by Engle (1993). Compared to other topics like model formulation and the estimate of continuous-time processes underlying option models, the specification of the loss function has not been given much consideration in the substantial and developing literature on option valuation.

For instance, the outstanding review of the literature by Campbell et al. (1997) omits any contributions addressing the significance of the choice of the loss function. Additionally, discussions of the loss function in option valuation that have already been made tend to focus on the statistical framework required to estimate the parameters of a hypothetical option valuation model, implicitly ignoring the influence the loss function has on the specification of the statistical model. The unspoken rule appears to be that, regardless of the purpose of any out-of-sample evaluation exercise, model parameters that are "correctly" estimated in-sample are automatically eligible for use. Contrarily, Christoffersen and Jacobs (2004) suggested that by applying the same loss function during estimate, one may typically reduce out-of-sample loss.

The realization that the selection of the loss function is in fact a component of the model specification serves as the basis for this advice. Therefore, using the same (statistical) models for estimation and evaluation seems logical. It is commonly acknowledged and frequently utilized in the literature that several loss functions can be used during the estimating and evaluating processes. Bakshi et al. (1997), for instance, used Dollar Mean Squared Error ($MSE) in estimates but also Percentage Mean Squared Error (%MSE) and $MSE in the evaluation stage, where $MSE stands for mean-squared absolute option pricing errors and %MSE for mean-squared relative option pricing errors. Mean squared absolute implied volatility error is referred to as Implied Volatility Mean Squared Error (IVMSE). Rosenberg and Engle (2002) use $MSE in estimation, but % hedging errors in evaluation, for example.

Hutchinson et al. (1994) evaluated the model outside of the sample using hedging mistakes in addition to an MSE-based option price divided by exercise price. Many papers (e.g., Renault, 1997; Jacquier and Jarrow, 2000) estimate model parameters from option prices using an estimation loss function based on the statistical characteristics of the underlying process or the statistical structure of the measurement errors, and then go on to evaluate the models outside of the sample using a different loss function. Pan (2002) uses the IVMSE for evaluation and the generalized method of moments (GMM) loss function for estimation. Chernov and Ghysels (2000) uses the efficient method of moments (EMM) to estimate parameters and the $MSE and %MSE loss functions to assess models.

EMM and \$MSE (normalized by the index value) are both used by Benzoni (2002) to estimate parameters, and \$MSE is then used to evaluate the model (again normalized). Finally, up until recently, many option valuation studies were carried out by estimating option model parameters from asset returns and inserting these parameters into option valuation formulas out-of-sample. However, most recent papers estimate option valuation parameters using option data or option data along with returns data. Once more, this relates to utilizing various loss functions both inside and outside of samples.

Comparing out-of-sample errors resulting from misaligned loss functions with errors from models where the in-sample and out-of-sample loss functions are the same could lead to issues. According to statistical literature, modifying the loss function is equivalent to adjusting the model specification (Granger, 1969; Engle, 1993). From this viewpoint, it is obvious that the "properly stated" model will produce the best in-sample fit but may or may not produce the greatest out-of-sample fit. An incorrectly specified model with precisely estimated parameters may perform better outside of the sample than an appropriately specified model.

Aligning the estimation and evaluation loss functions acts as a rule of thumb because there are no universal theorems that can direct authors in this situation. Therefore, there’s a need to be cautious to point out that while issues may occasionally develop when loss functions are out of alignment, they won't typically do so because the usefulness of this rule is an empirical topic. Dumas et al. (1998), for instance, compare the Practitioner Black-Scholes (PBS) model's out-of-sample performance to the out-of-sample performance of deterministic volatility models that were implemented with the same in- and out-of-sample loss functions. They conclude that the PBS model's valuation performance compares favorably to that of the deterministic volatility models.

The conclusions of Dumas et al. (1998) will therefore be strengthened if the PBS model is implemented properly because it is highly unlikely that doing so will cause the model's performance to decrease. But for the investigations of Heston and Nandi (2000) and Garcia et al. (2000), this might not be the case. Both of these publications employ the \$MSE loss functions for the out-of-sample comparison, but they estimate the PBS model using the implied volatility-based loss function. The PBS model and a GARCH model, both of which have identical in-sample and out-of-sample loss functions, are then contrasted by Heston and Nandi (2000).

According to Garcia et al. (2000), the PBS model outperforms the new Generalized Black-Scholes model, which is also constructed with aligned loss functions. The probable issue is that, even though both studies employ the PBS model as an evaluation benchmark, the benchmark's performance isn't as strong as it would be if it were implemented using the proper loss function.

Bakshi et al. (1997), Bams et al. (2009), Christoffersen and Jacobs (2004), Mikhailov and Nogel, (2003), and many others have used loss functions to estimate the parameters of the Heston model.

On that note, this dissertation has the objective to conduct a more thorough analysis of the loss function's effects on the Black-Scholes and Heston models.

Data Analysis

S&P 500 index options are used for the sample data because, as Rubinstein (1994) argues, this option market provides a context where the Black-Scholes condition seems most reasonably satisfied. Therefore, the sample data contains S&P 500 Index options traded on CBOE obtained on 1-May-2022 between 11:55 and 12:00. The reason behind using data that is not from the present period of this dissertation can be explained by the fact that 2022 data is prior to financial crisis resulting from the recent war. Therefore, extracted data wouldn't follow the normal flow from the financial markets and could cause misleading results. The Index Spot level registered at 11:55 of that day was 4,131.93 USD, and the dividend-yield registered at 11:55 of that day was 1.37 %. These options are European-style options and meet different expiration dates.

The cross-section data was obtained using Yahoo Finance and it includes the observed price, the spot price, the strike price, and the expiration date. It was also added a vector to distinguish between calls and puts (-1, 1 respectively), denominated VecPhi.

The remaining data needed to price options was the risk-free rate for each option maturity and the S&P 500 estimated dividend-yield. The dividend-yield was obtained using Bloomberg terminal. For the risk-free interest rate for each period, it was used the Nelson-Siegel-Svensson (NSS) approach to compute the USD yield curve. Therefore, the parameters used were taken from the FED's website on the 29th of April of 2022.

Thus, the parameters were applied on the next formula, for each corresponding maturity to the collected data to compute the risk-free interest rate:

$$r_c(T) = \beta_1 + \beta_2 * \left[\frac{1-\exp\left(-\frac{T}{\lambda_1}\right)}{\frac{T}{\lambda_1}}\right] + \beta_3 * \left[\frac{1-\exp\left(-\frac{T}{\lambda_1}\right)}{\frac{T}{\lambda_1}} - \exp\left(-\frac{T}{\lambda_1}\right)\right] + \beta_4 * \left[\frac{1-\exp\left(-\frac{T}{\lambda_2}\right)}{\frac{T}{\lambda_2}} - \exp\left(-\frac{T}{\lambda_2}\right)\right] \quad (1)$$

For the collected data, it was considered some exclusionary criteria as Dumas et al. (1998) suggest. First, were not taken into consideration options with fewer than six or more than hundred days to expiration. Finally, options deep ITM or deep OTM were also eliminated. Therefore, the criteria was the absolute moneyness $\left(\left|\frac{Strike\ price}{Spot} - 1\right|\right)$ being higher than 10 %, because options with these characteristics have small time premiums and hence contain close to zero information about the volatility function used on the option. On that note, 2 336 options were used within this dissertation. The table below shows the distribution of the collected data with the exclusionary criteria:

	DTM < 60	60 < DTM < 80	80 < DTM	All
Number of calls	736	266	166	1,168
Number of puts	736	266	166	1,168
Average Price	154.89	210	234.76	178.79
Average IV	0.2705	0.2626	0.2602	0.2672

Table 1 - Distribution of Collected Data

The goal is to use the collected data described before for the methodology and code/script developed to identify the most appropriate volatility function for Black-Scholes and Heston models, by minimizing the errors. To do so, it is imperative to first evaluate the goodness-of-fit of each model within the loss function used.

To access the quality of each model within the loss function, the following measurements were considered:

A. The root mean squared valuation error (RMSVE) that is the square root of the average squared deviations of the reported option prices from the model's theoretical values.
B. The F-test for the Black-Scholes model which refers to the global significance of the model, which means that if $p < 0.05$ the null is rejected and there is at least one coefficient that is statistically different from zero.
C. The Akaike (1973) Information Criterion (AIC) is calculated to appraise the potential degree of overfitting. The AIC is a statistical measure used to compare the relative performance of different pricing models. It takes into account both the goodness of fit and the complexity of the model. A lower AIC value suggests a better balance between model accuracy and complexity. The AIC penalizes the

goodness-of-fit as more degrees of freedom are added to the model in a manner similar to an adjusted R^2. Therefore, the R^2 it will be used instead. The highest value of the adjusted R^2 identifies the "best" model based on in-sample performance.

D. The RMSE - which is the most important criterion to fit if the main purpose of the model is prediction, is a statistical measure that quantifies the average difference between the model-implied prices and the observed market prices. It provides an overall assessment of the model's accuracy in pricing options. A lower RMSE indicates a better fit to the market data.

E. Out-of-Sample Testing: It is important to evaluate the models performance on data that was not used in the calibration process. This is known as out-of-sample testing and helps assess the model's ability to generalize to new data. By comparing the model's pricing accuracy on unseen data, practitioners can gain insights into its robustness and predictive power.

Methodology

Black-Scholes Model

As stated before, one of the weaknesses of the Black-Scholes model is the assumption of a constant volatility during the life of the option. The paper by Dumas et al. (1998), as well as many authors, for example Hull and White (1987), addressed the volatility of the underlying asset as not constant, and it is straight-forward to understand this issue. According to Dias (2022), “one possible explanation for the volatility skew found in equity option markets concerns leverage”. If the value from the company’s equity declines, the leverage (debt-equity ratio) from the company will increase, the equity becomes riskier, and, therefore, its volatility increases. Inversely, if the company’s equity increases in value, its volatility decreases. In summary, “it seems natural to expect the volatility of equity to be a decreasing function of price”.

The Black-Scholes model call option formula is given by,

$$c_t = S_t N(d_1) - X e^{-r\tau} N(d_2) \tag{2}$$

$$d_1 = \frac{\ln\left[\frac{S_t}{X}\right] + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}} \tag{3}$$

$$d_2 = d_1 - \sigma\sqrt{\tau} \tag{4}$$

where S is the price of the underlying asset at time t, X is the option's exercise price, τ is the expiry date, r is the risk-free interest rate, σ is the volatility rate, and N(d) is the cumulative unit normal density function with upper integral limit d. The interest rate on a T-bill with a comparable maturity is used for the risk-free rate. The implied volatility is calculated for each option price by finding the volatility rate that correlates the model price with the actual bid or ask quote.

Focusing on an improvement for the Black-Scholes model, we can introduce the model S, which “switches among the volatility functions given by models 1, 2 or 3, depending on whether the number of different option expiration dates in a given cross section is one, two or three, respectively”, presented by Dumas et al. (1998) as discussed before. Although the model S tries to approach Black-Scholes limitations, the dissertation will not focus on this model since it requires a big complexity for the calibration and computation.

Heston Model

The Heston model is a widely used stochastic volatility model in options pricing that addresses one of the key limitations of the Black-Scholes model – the assumption of constant volatility.

In reality, volatility is observed to be time-varying, and the Heston model introduces a stochastic process for the volatility parameter to capture this phenomenon. By allowing volatility to vary over time, the model can generate more accurate option prices that align with market observations. This is particularly valuable in pricing options with non-standard features, such as those with early exercise features, exotic options, or options on assets with time-varying volatility.

The incorporation of stochastic volatility in the Heston model provides several advantages. Firstly, it captures the empirically observed volatility clustering phenomenon, where high volatility tends to be followed by high volatility and vice versa. This feature is crucial for accurately pricing options during periods of market turbulence or sudden shifts in market sentiment. Secondly, the Heston model generates a volatility smile or skew, which matches market data more closely, reflecting the fact that options with different strike prices often have different implied volatilities.

In order to implement the Heston model, it is necessary to make several assumptions to simplify the dynamics of asset prices and volatility. These assumptions form the foundation of the model and provide a framework for option pricing and risk analysis. Therefore, the key assumptions typically include: constant interest rates – can be seen as a reasonable approximation for short-term options, hovewer it may not accurately capture the dynamics of interest rates for longer-term options; constant dividends; stochastic volatility process; independent asset price and volatility processes – in reality, there may be some correlation between asset prices and volatility, especially during times of market stress.

Although there are various numerical techniques available to calculate option prices under the Heston model, including Monte Carlo simulation, Fourier transform methods, and finite difference methods, this dissertation will not focus on the usage of these methdods because of the complexity and the computational intensity. Therefore, this dissertation will be focused on optimizing the parameters in order to minimize a heston calibration cost function in MATLAB. Therefore, the estimation of the parameters for the Heston model is crucial for accurately calibrating the model to market data and obtaining reliable option prices.

These parameters that need to be estimated include:

- v0 - The initial volatility level.
- Kappa - mean reversion rate: It determines how quickly the volatility reverts to its long-term average. A higher value implies faster mean reversion.
- Theta - long-term volatility: It represents the long-term average volatility level that the stochastic volatility tends to revert to. It is a measure of the mean level of volatility.
- Sigma - volatility of volatility: It represents the volatility of the volatility process. It quantifies the variability in the rate of change of the volatility.
- Rho - correlation: It represents the correlation between the Brownian motions driving the asset price and the volatility process. It measures the degree of correlation between the two processes.

Although initial guesses for the estimation of the parameters for the Heston model are not mandatory, it addresses the problem of the complexity of the calibration process and the fact that employes variety of optimizers. The optimal parameters may vary depending on the specific dataset and market conditions. Therefore, the chosen initial guesses follow the most commonly used initial guesses for the Heston model studies:

- v0 - average observed historical variances. For example, a value of 0.1 is often used as a starting point.
- Kappa - can be derived from the time scale over which volatility mean reverts. For example, a value of 1 or 2 is often used as a starting point.
- Theta - long-term average of the observed variances. For example, a value of 0.5 is often used as a starting point.
- Sigma - based on empirical observations or market knowledge. Values around 0.5 to 1.0 are often used as reasonable initial estimates.

- Rho - a moderate negative correlation between asset price and volatility. An initial guess of -0.5 or -0.7 is often used as a starting point.

There are various methods available for parameter estimation in the Heston model, with the Maximum Likelihood Estimation (MLE) being one of the most widely used approaches. The MLE method seeks to find the parameter values that maximize the likelihood of the observed option prices given the model. These parameters are essential in the Heston model for option pricing as they govern the dynamics of the underlying asset's volatility. They can be estimated from historical data or calibrated using option prices through various techniques, such as maximum likelihood estimation or optimization methods. In this dissertation, it was used the historical data for the parameter estimation.

Calibrating the Heston model to market data is an essential step in option pricing. The calibration process involves adjusting the model parameters to minimize the difference between the prices predicted by the Heston model and the observed market prices. By achieving a close match between the model-implied prices and the market prices, the calibrated model can accurately capture the market dynamics and provide reliable option valuations. The calibration technique used requires an objective function that quantifies the difference between the model prices and the market prices. Therefore, minimizing the root mean squared error will be the main focus of this objective function in order to find the parameter values that provide the best fit to the market data.

Double Heston Model

The double Heston model is an extension of the Heston model that incorporates two stochastic volatility processes to capture additional features and complexities of the underlying asset. It is a popular model in quantitative finance and is used to price and hedge exotic options and structured products. Like the Heston model, the double Heston model also has some assumptions:

- Constant interest rates: The model assumes a constant risk-free interest rate throughout the option's life.
- Constant dividends: It assumes a constant dividend yield for the underlying asset, which represents the periodic cash flows received by the option holder.

The option pricing in the double Heston model is typically done using numerical methods such as Monte Carlo simulation or finite-difference methods. These methods involve simulating paths for the asset price and the two volatility processes and calculating the option's payoff under each simulated path.

The double Heston model provides a flexible framework for pricing complex options, especially those with features that cannot be accurately captured by a single volatility process. By incorporating two stochastic volatility processes, it can capture more intricate volatility dynamics and improve the accuracy of option pricing. However, it is important to note that the model's increased complexity also requires careful calibration and validation to ensure its suitability for specific applications and market conditions.

On that note, the double Heston model offers several advantages when comparing to the standard Heston model such as:

- Improved Volatility Surface Fitting: The double Heston model introduces an additional stochastic volatility process, allowing for greater flexibility in capturing complex volatility dynamics. The Heston model has a single volatility process, which may not be sufficient to accurately capture certain market behaviors, such as volatility smiles or skewness.
- Improved Calibration: The Heston model calibration can sometimes be challenging due to the interplay between the model parameters. The additional flexibility of the double Heston model allows for better calibration to market data. By incorporating the second volatility process, the model can more effectively match observed option prices. This is particularly important in risk management and pricing of options that are sensitive to extreme events, such as barrier options or options with path-dependent features.

Although double Heston model is an improved version of the Heston model when it comes to option pricing, it also has its limitations when compared to other option pricing models, for example:

- Complexity and Computational Intensity: The double Heston model is more complex compared to simpler option pricing models like the Black-Scholes model or even the standard Heston model. This increased complexity can lead to higher computational requirements and longer computation times, especially when pricing exotic options or performing extensive simulations.

- Calibration Challenges: with more parameters to estimate, the calibration may become more challenging, and there is a risk of overfitting the model to the historical data. Robust calibration techniques and careful validation are essential to ensure reliable parameter estimation.
- Simplicity of Other Models: While double Heston model incorporates additional features and complexities through two stochastic volatility processes, there may be cases where simpler models can provide comparable pricing accuracy.

Although the double Heston model can be more accurate for some cases of option pricing, its complexity and calibration challenges make this model less appealing to use. Therefore, only the Black-Scholes model and the Heston model will be taken into consideration for option pricing.

Deterministic Volatility Functions

The behavior of the volatility on options is observable when computing the implied volatility since, usually, ATM options show a lower value of volatility, comparing with deep OTM calls and deep ITM puts.

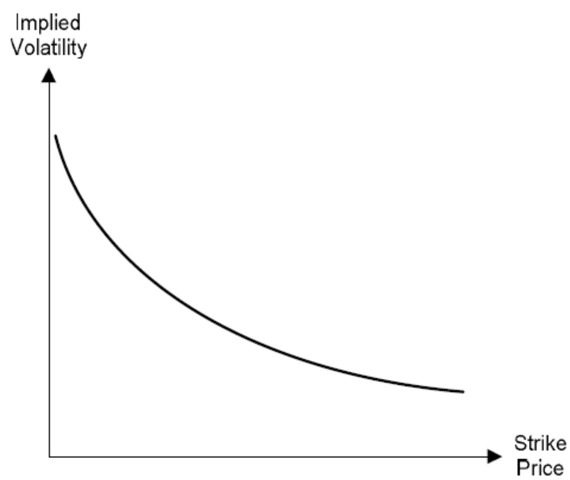

Figure 1 - Volatility Smile

The plot above relates the implied volatility observed with the time adjusted moneyness (TAM). Negative values of the TAM match with ITM puts and OTM calls, which suggests that the implied volatility is not constant, showing some dependency on the Tau (time to expiration) and/or on the strike price. For that matter, it is possible to conclude that a Deterministic Volatility Function depends on those two specifications from an option.

On that note, the code/script will be developed to compute and determine which functional form of the model is the best fitted volatility model. These computations are going to be made using a regression model, assuming the implied volatility, computed from collected data, as the dependent variable and using the strike price and the Tau as explanatory variables. The parabolic shape of the volatility smile, and its dependence on moneyness and time to maturity, has motivated researchers to model implied volatility as a quadratic function, since the dependency between the variables is not linear.

Dumas et al. (1998) names this approach as the deterministic volatility function for modeling the implied volatility, considering four different volatility specifications for the DVF:

Model 0: $\sigma_{IV} = \max(0.01, a_0)$ (5)

Model 1: $\sigma_{IV} = \max(0.01, a_0 + a_1 . K + a_2 . K^2)$ (6)

Model 2: $\sigma_{IV} = \max(0.01, a_0 + a_1 .K + a_2 .K^2 + a_3 .T + a_4 . K . T)$ (7)

Model 3: $\sigma_{IV} = \max(0.01, a_0 + a_1 .K + a_2 .K^2 + a_3 .T + a_4 . T^2 + a_5 . K . T)$ (8)

where σ_{IV} is the Black-Scholes model (BSM) implied volatility, K is the strike price, T is the time to maturity, and a_i are the model parameters, with i ∈ {1,2,3,4,5}. Observing these models, it is possible to take some conclusions regarding the volatility:

Model 0: Assumes constant volatility with no dependence on the strike price or the time to maturity, thus corresponding to the BSM model.

Model 1: Attempts to capture the variation in the volatility with a quadratic function of the strike price and with no dependency on time to maturity.

Model 2: Adds a dependence on time to maturity when compared to the model 1 and assumes an interaction between moneyness and time to maturity in its last term.

Model 3: Allows for the relationship between volatility and time to maturity to be quadratic also, with the objective of capturing additional variation due to the time to maturity of the option.

It can also be concluded that all models have a threshold value of 0.01 in order to avoid possible negative values of fitted volatility.

Loss Functions

Loss Functions – Black-Scholes Model

As it was said before, different purposes, for example, market manipulation, speculating, or hedging, imply different loss functions for the model errors. Since the existing academic literature tend to ignore the evaluation loss function when estimating the parameters, this dissertation will also have a focus on this.

As well as in the paper from Christoffersen and Jacobs (2004), the focus will not be on a specific loss function but rather in testing the different loss functions presented by these authors and taking some conclusions regarding the collected data and option valuation. For that matter, it was used in-sample and out-of-sample fits, since a miss specified model with precisely estimated parameters (out-of-sample) may outperform the correctly specified model. In that way, the out-of-sample data was calculated to 1-day, 5-days, and 20-days out-of-sample, by adding 1, 5 and 20 days and divide by 365 that corresponds to the number of days from a year to the time-to-maturity from collected data.

Christoffersen and Jacobs (2004) identify three different loss functions at the estimation and evaluation stages:

IV MSE: Stands for mean squared absolute implied volatility error and it is defined as:

$$IV\ MSE(\theta) \equiv \frac{1}{n}\sum_{i=1}^{n}(\sigma_i - \sigma_i(\theta))^2 \tag{9}$$

where the implied volatilities are:

$\sigma_i = BS^{-1}(C_i, T_i, X_i, S, r)$ and $\sigma_i(\theta) = BS^{-1}(C_i(\theta), T_i, X_i, S, r)$,

and BS^{-1} is the inverse of the Black-Scholes formula, T_i is the time-to-maturity, X_i is the strike price, S is the price of the underlying stock, and r is the riskless interest rate.

$ MSE: Stands for mean squared dollar errors and it is defined as:

$$\$\ \mathrm{MSE}(\theta) = \frac{1}{n}\sum_{i=1}^{n}(C_i - C_i(\theta))^2 \tag{10}$$

where C_i and $C_i(\theta)$ are the data and model option prices, respectively, and n is the number of option contracts used. This loss function has the advantage that the errors are easily interpreted as $-errors once the square root is taken of the mean-squared error. "However, the relatively wide range of option prices across moneyness and maturity raises the problem of heteroskedasticity for $MSE-based parameter estimation" Christoffersen and Jacobs (2004).

% MSE: Stands for percent mean-squared error and it is defined as:

$$\%\ \mathrm{MSE}(\theta) = \frac{1}{n}\sum_{i=1}^{n}((C_i - C_i(\theta))/C_i)^2 \tag{11}$$

where the % sign is a convenient measure for the relative loss. In fact, it is not multiplied by 100, therefore the losses are not expressed in percent but rather in decimals.

This loss function "has the advantage that a $1 error on a $50 dollar option carries less weight than a $1 error on a $5 option, which is sensible from a rate-of-return perspective" Christoffersen and Jacobs (2004). But the fact that the short time-to-maturity out-of-money options with valuations close to zero will implicitly get assigned a lot of weight and thus create a numerical instability, can be seen as a disadvantage to this loss function.

Some researchers instead favor the relative or percent mean-squared error loss function because the $ MSE loss function implicitly assigns a lot of weight to options with high valuations and, thus, high $-errors.

As mentioned before, there are other estimation loss functions that are used in the literature. "Functions based on hedging or speculation loss could potentially be more interesting" Christoffersen and Jacobs (2004), but the focus will be on these three functions listed before since they are the most suitable to this kind of research.

Christoffersen et al. (2004) stated that the estimation of the loss function can be done by using the simple model possible, the PBS model, which is implemented in three steps. First, the implied volatility is calculated for each observed option. Secondly, the implied volatilities are regressed on different polynomials, using simple ordinary least squares (OLS) for the IV MSE loss function. Finally, the fitted values for volatility are plugged back into the Black-Scholes formula with the objective of obtaining the estimated model price. As seen before, the estimation loss function, defined on implied volatilities, is different from the evaluation loss functions, defined on percent or dollar pricing errors. For that reason, when the evaluation loss function is $ MSE or is % MSE, the use of the non-linear least squares (NLS) to directly estimate θ is the appropriate procedure.

Loss Functions – Heston Model

As was previously mentioned, the Heston model's parameters are estimated using loss functions to bring quoted option prices as closely as possible to model option values. Volatilities that are quoted and predicted by models can also be used in place of prices. This strategy makes use of the discrepancy between model prices or indicated volatilities and quoted market prices. In order to make the model prices as near to market prices as possible, these estimates are values that minimize the value of the loss function. On that note, it is necessary to use a restricted minimization technique in order to respect the parameter limitations:

$$k > 0\ ;\ \theta > 0\ ;\ \sigma > 0\ ;\ v_0 > 0\ ;\ \rho \in [-1,1]$$

Loss functions generate estimates of the risk-neutral Heston model parameters because they employ market option prices (or implied volatility derived from those prices) as inputs. Assuming that there's a set of N_k strikes $K_k\ (k = 1, \dots, N_k\ )$ and a set of N_T maturities $T_t\ (t = 1, \dots, N_T\ )$. Then, the market price $C(T_t, K_k) = C_{tk}$ and a corresponding model price $C(T_t, K_k;\ \Theta\ ) = C_{tk}^{\Theta}$ produced by the Heston model for each maturity-strike combination (T_t, K_k). Each option has an optional weight w_{tk} attached to it.

Those loss functions that reduce the difference between quoted and model prices fall under the first group of loss functions. The error is typically expressed as the squared difference between the model price and the quoted price, though relative errors may also be used. When applying the mean error sum of squares (MSE) loss function, parameter estimates are generated by minimizing with respect to Θ, where N is the number of quotes.

$$\frac{1}{N}\sum_{t,k}^{n} w_{tk}\ (C_{tk} - C_{tk}^{\Theta})^2 \qquad (12)$$

The formula described above is a loss function that is used to estimate the RMSE parameter.

$$\frac{1}{N}\sum_{t,k}^{n} \frac{w_{tk}\ (C_{tk} - C_{tk}^{\Theta})^2}{C_{tk}} \qquad (13)$$

As an alternative, one might describe the mistake in terms of the absolute value, which would result in $\left|C_{tk} - C_{tk}^{\Theta}\right|$, and establish a loss function as in Equations (12) and (13).

The fact that short maturity, deep OTM options with minimal value contribute little to the sum in the MSE loss function is one of its well-known drawbacks (12). As a result, the optimization will typically favor ITM, long-maturity options at the expense of other options. One solution is to only utilize ITM options, in which case call options would be used for strikes below the spot price and put options would be used for strikes above the spot price in (12). Utilizing the RMSE loss function is the other solution (13). But with RMSE, the reverse result happens, which is a problem.

In fact, because C_{tk} is in the denominator, options with low market value will significantly increase the amount in (13).

However, the over-and under-contribution can be reduced by giving each component in the objective function a weight, though the weights are often chosen subjectively. Those loss functions that reduce the discrepancy between quoted and model suggested volatilities fall under the second group of loss functions. Once more, the error is typically described as the squared difference, absolute difference, or relative difference between the implied volatilities in the model and those in the quoted price. Since options are frequently quoted in terms of implied volatility and since the model's fit is frequently evaluated by comparing quoted and model implied volatilities, this kind of loss function makes sense.

Therefore, for instance, the loss function is used to estimate the IVMSE parameter.

$$\frac{1}{N}\sum_{t,k}^{n} w_{tk}\,(IV_{tk} - IV_{tk}^{\Theta})^2 \tag{14}$$

where $IV(T_t, K_k) = IV_{tk}$ and $IV(T_t, K_k;\ \Theta) = IV_{tk}^{\Theta}$ are the quoted and model implied volatilities, respectively. The relative and absolute versions can also be used like on the other loss functions described above.

Equation (14) primary drawback is that it requires a lot of numerical computation. To generate the quantity $(IV_{tk} - IV_{tk}^{\Theta})^2$, one must first collect each Heston price C_{tk} at each iteration of the optimization. From there, one must use a root-finding procedure, such as the bisection algorithm, to extract the implied volatility IV_{tk} from C_{tk}. One solution is to substitute IV_{tk} with the approximated implied volatility from Lewis (2000) Series II extension. This allows to completely avoid the bisection algorithm. Using the loss function presented in Christoffersen et al. (2009), which approximates the IVMSE in Christoffersen and Jacobs (2004), is another way of solving this problem.

It uses the reciprocal of the squared Black-Scholes Vega as the weight in (16). Therefore, the parameter estimates from their method are based on the loss function.

$$\frac{1}{N}\sum_{t,k}^{n}\frac{(C_{tk}-\ C_{tk}^{\Theta})^2}{BSVega_{tk}^2} \qquad (15)$$

where $BSVega_{tk}$ is the sensitivity of the Black-Scholes option price with respect to the market implied volatility IV_{tk}, evaluated at the strike $K_k$and maturity T_k, that is

$$BSVega_{tk} = Se^{-q^T}n(d_{tk})\sqrt{T} \quad , \qquad (16)$$

with

$$d_{tk} = \frac{\ln\left[\frac{S}{K_t}\right] + \left(r - q + 0.5\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad (17)$$

and where $n(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\Pi}}$ is the standard normal density.

Code/Script Steps

Black-Scholes Model – IVMSE Loss Function

For the first part, the objective was to minimize the IV RMSE using the ordinary least squares (OLS), for each DVF and compare which is the best model. To do so, it was used the function *fitlm* which returns a linear regression model of the dependent variable, in this case the implied volatility computed before, fit to the explanatory variables, corresponding to the strike price and time to expiration.

Given these three linear regressions, it was made a comparison between the models, using the p-values and the adjusted R^2 (is a better model evaluator when compared to the R^2 because it correlates the variables more efficiently). Thus, only the linear regressions with p-value higher than 0.05 are not rejected, and the model with higher adjusted R^2 is chosen as the best fitted model to explain the volatility from the options on the collected data.

Having the best fitted model, the next step from the code/script was to compute the volatility, considering either the model 1, 2 or 3 from the DVF corresponding to the chosen model. As it was said before, it was also considered the criteria of the maximum between 0.01 and the volatility computed using the best fitted model. The next step had the intention of calculating the option prices, using the *BSM_price*, considering the implied volatility computed before and the implied volatility from the best fitted model.

Therefore, it was possible to compare the difference obtained from the theoretical prices computed by the Black-Scholes model formula, using the implied volatility and the volatility from the best fitted model, with the bid-ask mid prices observed in the market. These valuation errors were compared between each other, returning a final vector for the lower values when summing the valuation error.

It was used the loss function IV MSE to the model's evaluation performance in-sample and on the next step the focus it to make out-of-sample (1-Day, 5-Days, 20-Days) simulations for the best fitted model, from the DVF. In that way, it was added 1/365, 5/365 and 20/365 years to the time to expiration from the collected data, and therefore were created three different vectors for each new maturity. Since the maturities are different, it was needed to compute the implied volatility from this out-of-sample data, using the function *ImpVolBSM*, considering the threshold value of 0.01. In the next step, it was computed three different linear regression models, using the implied volatility for each maturity as dependent variable.

The final focus from the part 1 was to compare the RMSE values from the best fitted model and the regressions from out-of-sample data to understand which is the model that best fits the observed data and better explains the volatility within the collected data.

Black-Scholes Model – $ MSE Loss Function

For the second part, the objective was to minimize the $ RMSE using the non-linear least squares (NLS), for each DVF and compare which is the best model. To do so, it was used the function *nlinfit* which returns a non-linear least-squares estimates of the parameters of a non-linear model. The dependent variable, in this case is the observed price from collected data, and the explanatory variables, corresponding to the strike price and time to expiration.

To use the NLS in MATLAB it is necessary to determine a model function, which in this case was the formula from the BSM model adjusted to the sigma (volatility) being defined as either the model 1, 2 or 3 from the DVF. It was also needed to make some initial guesses for the beta values about to be estimated. Considering the initial guesses as zeros, the code/script returned the estimated betas, residuals, jacobian matrix, estimated variance-covariance for the estimated coefficients, estimate of the variance of the error term, defined as MSE, and a structure containing details about the error model.

After computing the NLS for each model from the DVF, it was computed the RMSE which corresponds to the square root of the MSE, calculated before. As it was done in part 1, the models from the regressions were compared and one was considered as the best fitted model, although the measure of comparison for this regression is the RMSE since both the p-value and the adjusted R^2 are invalid measures for the NLS. In that way, the best fitted model will be the one with the lowest RMSE.

Similarly, to the first part, it was computed the volatility for the chosen model, using the estimates for the betas regressed before and the estimated option prices using these values for the volatility. The estimated option prices using the implied volatility were also computed, as well as the valuation errors between the estimated prices from the implied volatility and the volatility from the best fitted model, and the code/script compares both and returns the best as the final valuation error computed.

Until this step it was used the loss function $ MSE to the model's evaluation performance in-sample and on the next step the focus was to make out-of-sample (1-Day, 5-Days, 20-Days) simulations for the best fitted model from the DVF. Therefore, the code/script was developed with the same logic from part 1, with the exception that the regressions were made with the NLS and, consequently, considering different times to expiration on the function model for this regression.

After computing these three new regressions, it was again computed the RMSE for each model, in order to be able to compare the results with the RMSE from the best fitted model from the DVF. Thus, the main and final focus from the second part was to compare the RMSE from all the models in order to understand which is the model that best fits the observed data and better explains the volatility within the collected data.

Black-Scholes Model – % MSE Loss Function

For the third part, the objective was to minimize the % RMSE using the non-linear least squares (NLS), for each DVF and compare which is the best model. To do so, it was used the function *nlinfit* which returns a non-linear least-squares estimates of the parameters of a non-linear model. The dependent variable, in this case is considered as being a vector of ones, and the explanatory variables, correspond to the strike price and time to expiration. Therefore, the model function used is given by:

$$model\ function = \frac{C_i - C_i(\theta)}{C_i} = 1 - \frac{C_i(\theta)}{C_i} \tag{18}$$

As it was said before, to use the NLS in MATLAB it is necessary to determine a model function, which in this case was the formula from the BSM model adjusted to the sigma (volatility) being defined as either the model 1, 2 or 3 from the DVF, divided by the observed prices from the collected data. It was also needed to make some initial guesses for the beta values about to be estimated. Considering the initial guesses as zeros, the code/script returned the same measures as on the previous part.

The rest of the steps are very similar to the ones made for the second part, applying the same formulas, and using the same logic but with the exception that the regressions were made with a different function model for the NLS.

Heston Model – IVMSE Loss Function

For the first part, the objective was to minimize the parameters of the Heston model for the Implied Volatility Loss Function using an optimization function, for each DVF and compare which is the best model. To do so, it was used the function *fmincon* which returns optimized values for the Heston model parameters, in order to reduce the discrepancy between the model-implied prices and the market prices. The process continues until a satisfactory level of fit is achieved.

Regarding the initial parameters, it was necessary to make some initial guesses for the initial variance, kappa, theta, sigma, and rho, as it was mentioned before. Therefore, the initial guesses used are shown at the table below:

v0	Kappa	Theta	Sigma	Rho
0.1	1	0.5	0.8	-0.6

Table 2 - Initial Values for Heston Model Parameters

As it was mentioned before, even though the MLE estimation for the parameters could have been used to optimize the IVMSE Loss function, it is believed that it could result in a worst optimization rather of an improvement. Even though it was not applicated, it is important to mention that to use this approach it is necessary to define a likelihood function based on the Heston model and use a suitable optimization algorithm.

The next objective was to minimize the IV RMSE using the ordinary least squares (OLS) like it was used on the BSM model, for each DVF and compare which is the best model. To do so, it was used the function *fitlm* which returns a linear regression model of the dependent variable, in this case the implied volatility computed before, fit to the explanatory variables, corresponding to the strike price and time to expiration.

Given these three linear regressions, it was made a comparison between the models, using the p-values and the adjusted R^2 (is a better model evaluator when compared to the R^2 because it correlates the variables more efficiently). Thus, only the linear regressions with p-value higher than 0.05 are not rejected, and the model with higher adjusted R^2 is chosen as the best fitted model to explain the volatility from the options on the collected data.

Therefore, it was necessary to define the DVF and the initial parameters for the IVMSE Loss Function optimization to be completed. These four models include the model 1, 2 or 3 from the DVF mentioned before and the model that is given by the implied volatility, computed by using the Heston model option pricing.

Having the best fitted model, the next step from the code/script was to compute the volatility, considering either the model 1, 2 or 3 from the DVF corresponding to the chosen model. The next step had the intention of calculating the option prices, using the *Heston_price*, considering the implied volatility computed before and the implied volatility from the best fitted model.

Therefore, it was possible to compare the difference obtained from the theoretical prices computed by the Heston model formula, using the implied volatility and the volatility from the best fitted model, with the bid-ask mid prices observed in the market. These valuation errors were compared between each other, returning a final vector for the lower values when summing the valuation error.

It was used the loss function IV MSE to the model's evaluation performance in-sample and on the next step the focus it to make out-of-sample (1-Day, 5-Days, 20-Days) simulations for the best fitted model, from the DVF. In that way, as it was done on the Black-Scholes code/script, it was added 1/365, 5/365 and 20/365 years to the time to expiration from the collected data, and therefore were created three different vectors for each new maturity. Since the maturities are different, it was needed to compute the option prices from this out-of-sample data, using again the function *Heston_price*.

The final focus was to compute and compare the RMSE values from the best fitted model, the implied volatility, and out-of-sample option prices to understand which is the model that best fits the observed data and better explains the option pricing within the collected data, which will be the model with the lowest RMSE.

Heston Model – $ MSE Loss Function

For the second part, the objective was also to minimize the parameters of the Heston Model for the Dollar Loss Function using an optimization function, for each DVF and compare which is the best model. To do so, it was also used the function *fmincon* which returns optimized values for the Heston model parameters, in order to reduce the discrepancy between the model-implied prices and the market prices. The initial parameters were the same as the ones used on part I.

For the second part, the objective was to minimize the $ RMSE using the non-linear least squares (NLS), for each DVF and compare which is the best model. To use the NLS in MATLAB it is necessary to determine a model function, which in this case was the formula from the Heston model adjusted to the parameters.

Therefore, the code/script was developed with the same logic from Part I for the initial parameters and the DVF, as well as the verification of the best-fitted model and the out-of-sample simulations but applying it for the Dollar MSE Loss Function.

After computing all option prices for the different simulations, it was again computed the RMSE for each model, in order to understand which is the model that best fits the observed data and better explains the option pricing within the collected data, which will be the model with the lowest RMSE once again.

Heston Model – % MSE Loss Function

For the third part, the objective was also to minimize the parameters of the Heston model for the Percentage Dollar Loss Function using an optimization function, for each DVF and compare which is the best model. To do so, it was also used the function *fmincon* which returns optimized values for the Heston model parameters, in order to reduce the discrepancy between the model-implied prices and the market prices. The initial parameters were the same as the ones used both on part I and part II.

The rest of the steps are very similar to the ones made for the second part, applying the same formulas, and using the same logic but with the exception that all computations and simulations were done by applying for the Percentage Dollar MSE Loss Function.

Estimation Results

Black-Scholes Model

Regarding the values from the regressed models for the F-test p-values, it is necessary to consider that the number of observations can have an impact on the real distribution. If there is bias, more observations would have a negative impact but if there is conformity with the real data, more observations improve the F-test. Usually, more observations reduce the p-values as it optimizes the calibration of the parameters. It is similar to identify the probability of the population as random and not a good sample to use, to test the DVF.

Nowadays, the markets are not flowing according to the past years, since there are many sources of uncertainty at the moment, having an immediate impact on the option prices. Thus, since the market is very volatile these days, there can be very specifications that impact the collected data as being an outlier per say. Therefore, the usage of the DVF may not be the best way to determine the implied volatility and evaluate option prices, under the BSM model.

IVMSE Loss Function

Regarding part 1, that corresponds to the IVMSE loss function, all the p-values from the F-statistic are below 0.05, which means that there is at least one coefficient that is statistically different from zero on all models from the DVF. Therefore, all models were also compared, taking the adjusted R^2 into account, with the code/script returning the best fitted model as the one with the highest value. Thus, the model chosen (best fitted model) within the DVF, for in-sample data represents the model 3 and, consequently it will be the model used on out-of-sample analysis.

The next step, and the focus of the dissertation, was to compare the RMSE between the best fitted model, the 1-day out-of-sample model, the 5-days out-of-sample model, and the 20-days out-of-sample model. Looking to table 2, presented below, it is possible to understand that the best fitted model (BFM) represents the model that best fits the collected data and has a better performance to explain the observed prices, for the first loss function. The model given by 20-days out-of-sample is the one that performs worst, taking this measure into consideration.

RMSE Outputs from Part 1	
RMSE from BFM	0.2838
RMSE from OFS 1 Day	0.2842
RMSE from OFS 5 Days	0.2862
RMSE from OFS 20 Days	0.2923

Table 3 - RMSE Outputs from Black-Scholes Model Part I

This can be explained by the fact that the third model of the DVF includes additional terms, tending to provide a better fit to the observed prices in the in-sample dataset, meaning that it can capture nuances and variations more effectively in the historical data. In the case of the 20-days out-of-sample scenario, the market conditions on underlying factors may change over time and the third DVF, being more complex, may not adapt well to these changes resulting in less accurate predictions, meaning that it may be overfitting to the historical data.

In the paper presented by Dumas et al. (1998), the model S is the model that has a better performance to explain the observed prices.

$ MSE Loss Function

Regarding part 2, that corresponds to the $ MSE loss function, and since the p-values from the F-statistic and the adjusted R^2 are not measures used on the NLS regression, the comparison between models was made using the MSE measure. Thus, the model chosen within the DVF, for in-sample data represents the model 1 and, consequently it will be the model used on the out-of-sample analysis. Comparing the valuation error between the prices given by the implied volatility from model 0 and the best fitted model with the observed prices on the collected data, it was possible to understand that the model 1 from the DVF has the best results for this measure.

Again, the next step, and the main focus of the dissertation was to compare the RMSE between the best fitted model, the 1-day out-of-sample model, the 5-days out-of-sample model and the 20-days out-of-sample model. Looking to table 3, presented below, it is possible to understand that the model, given by 20-days out-of-sample data, represents the model that best fits the collected data and has a better performance to explain the observed prices, for the second loss function. Curiously, the supposedly best fitted model (model 1) is the one that performs worst, taking this measure into consideration.

RMSE Outputs from Part 2	
RMSE from BFM	0.3955
RMSE from OFS 1 Day	0.3951
RMSE from OFS 5 Days	0.3938
RMSE from OFS 20 Days	0.3889

Table 4 - RMSE Outputs from Black-Scholes Model Part II

Even though the first model of the DVF might have achieved the best fit to the in-sample data, this strong fit might also have resulted from overfitting the model to the noise or random fluctuations in the training data, meaning that the first DVF might have captured the historical data's idiosyncrasies rather that the underlying market dynamics. On the other hand, simpler models tend to have higher bias but lower variance, which means they may generalize better to new, unseen data. In the case of the 20-days out-of-sample scenario, market conditions and underlying factors can change significantly and the first DVF might not be flexible enough to adapt to these changes because of its simplicity, leading to large prediction errors. This highlights the importance of cross-validation and out-of-sample testing to assess a model's true predictive performance.

For the paper presented by Christoffersen and Jacobs (2004) the best fitted model from the DVF is the one that has a better performance to explain the observed prices.

% MSE Loss Function

Regarding part 3, that corresponds to the % MSE loss function, once again the p-values from the F-statistic and the adjusted R^2 are not measures used on the NLS regression. Therefore, the comparison between models was also done using the RMSE measure. Thus, the model chosen within the DVF, for in-sample data represents the model 1 and, consequently it will be the model used on out-of-sample analysis. Comparing the valuation error between the prices given by the implied volatility from model 0 and the best fitted model with the observed prices on the collected data, it was possible to understand that the model 0 from the DVF has the best results for this measure.

The next step, and again the main focus of the dissertation was to compare the RMSE between the best fitted model, the 1-day out-of-sample model, the 5-days out-of-sample model and the 20-days out-of-sample model. Looking to table 4, presented below, it is possible to understand that the best fitted model (model 1) represents the model that best fits the collected data and has a better performance to explain the observed prices, for the third loss function. Curiously, all models present very close results and the model, given by 20-days out-of-sample data, is the one that performs worst, taking this measure into consideration.

RMSE Outputs from Part 3	
RMSE from BFM	0.43483
RMSE from OFS 1 Day	0.43484
RMSE from OFS 5 Days	0.43485
RMSE from OFS 20 Days	0.43487

Table 5 - RMSE Outputs from Black-Scholes Model Part III

This can be explained by the fact that the first model of the DVF tends to have higher bias, which means it can generalize better to different market conditions. In the case of the 20-days out-of-sample scenario, the market conditions on underlying factors can change significantly and the first DVF might not be flexible enough to adapt to these changes because of its simplicity, leading to larger prediction errors.

For the paper presented by Christoffersen and Jacobs (2004) the best fitted model from the DVF is also the one that has a better performance to explain the observed prices and the 20-days out-of-sample model is also the one that performs worst.

Heston Model

Since there are numerous causes of uncertainty at the present, which have a direct impact on option pricing, the markets are not currently operating in a manner consistent with previous years. As a result, given how unpredictable the market is right now, there may be several factors that cause the data to be obtained to be an outlier. Considering this, using the DVF in the Heston model to calculate the option prices also may not be the best course of action.

IVMSE Loss Function

Regarding part 1, that corresponds to the IVMSE loss function, all the p-values from the F-statistic are below 0.05, which means that there is at least one coefficient that is statistically different from zero on all models from the DVF. Therefore, all models were also compared, taking the adjusted R^2 into account, with the code/script returning the best fitted model as the one with the highest value. Thus, the model chosen (best fitted model) within the DVF, for in-sample data represents the model 3 and, consequently it will be the model used on out-of-sample analysis.

Comparing the RMSE between the best fitted model, the 1-day out-of-sample model, the 5-days out-of-sample model, and the 20-days out-of-sample model was the following stage and the main emphasis of the dissertation. The 20-days out-of-sample model is the one that best fits the collected data and performs better to explain the observed prices, according to table 5, which is shown below. This is true for the first loss function. Curiously, the model that performs the worst when this metric is considered is the one provided by the BFM.

RMSE Outputs from Part 1	
RMSE from BFM	0.3415
RMSE from OFS 1 Day	0.3098
RMSE from OFS 5 Days	0.3188
RMSE from OFS 20 Days	0.2666

Table 6 - RMSE Outputs from Heston Model Part I

Although the BFM represents the best-fitted model among the candidate models based on the calibration process, it may not necessarily have the lowest IVMSE, as calibration aims to fit option prices, not necessarily implied volatilities. Thus, even though it’s the best-fitted model in terms of options prices, it might not be the best at reproducing implied volatilities, leading to a higher IVMSE. Therefore, the BFM may have overfitted the historical data, meaning that the model captures noise in the data rather than the underlying patterns. The third model is more complex which can lead to more flexibility in fitting historical data. Therefore, considering that financial markets are dynamic, and implied volatilities can change over time due to various factor, the BFM may not capture changes in implied volatilities as effectively as a model specifically designed for forecasting over longer time horizons, such as the 20-days out-of-sample scenario.

For the paper presented by Christoffersen and Jacobs (2004) the best fitted model from the DVF is the one that has a better performance to explain the observed prices.

$ MSE Loss Function

Regarding part 2, that corresponds to the $ MSE loss function, the comparison between models was done using the RMSE measure. Thus, the model chosen within the DVF, for in-sample data represents the model 1 and, consequently it will be the model used on out-of-sample analysis. It was feasible to determine that the model 1 from the DVF has the best results for this measure by comparing the valuation error between the prices indicated by the implied volatility from model 0 and the best fitted model with the observed prices on the collected data.

Once more, the following step and the primary objective of the dissertation was to compare the RMSE between the best fitted model, the 1-day out-of-sample model, the 5-days out-of-sample model, and the 20-days out-of-sample model. By examining table 6, which is provided below, it is feasible to comprehend that the model provided by the BFM reflects the one that best fits the data that was gathered and has a better ability to explain the observed prices, for the second loss function. The 20-days out-of-sample model is the one that performs the worst.

RMSE Outputs from Part 2	
RMSE from BFM	0.5224
RMSE from OFS 1 Day	0.5305
RMSE from OFS 5 Days	0.5650
RMSE from OFS 20 Days	0.7015

Table 7 - RMSE Outputs from Heston Model Part II

This can be explained because the BFM may have overfitted the historical data, meaning that it captures noise in the data rather than the underlying patterns. Over fit models can perform exceptionally well on the calibration data but generalize poorly to new, out-of-sample data. This is a common issue when choosing models based solely on calibration performance. On this case, the first DVF is a simpler model with fewer parameters which may be less prone to overfitting but may also have limited flexibility to capture complex patterns in the data. Therefore, the BFM's superior performance in terms of $ MSE on historical data could be due to its calibration to this specific dataset. However, this advantage may not necessarily extend to out-of-sample data, specifically if the DVF is not that complex. On that note, the 20-days out-of-sample while performing poorly in the calibration dataset, could be better to forecasting longer periods and could avoid overfitting issues if the purpose was to choose a model for financial forecasting.

For the paper presented by Christoffersen and Jacobs (2004) the best fitted model from the DVF is the one that has a better performance to explain the observed prices, therefore following the same results as the first loss function computed before.

% MSE Loss Function

The RMSE metric was also used to compare the models in relation to part 3, which corresponds to the % MSE loss function. As a result, the model selected within the DVF to describe in-sample data is model 1, and as a result, it will be the model utilized in out-of-sample analysis. It was feasible to determine that the model 1 from the DVF has the best results for this measure by comparing the valuation error between the prices indicated by the implied volatility from model 0 and the best fitted model with the observed prices on the collected data.

The dissertation 's following stage was comparing the RMSE of the best fitted model with that of the 1-day out-of-sample model, the 5-day out-of-sample model, and the 20-day out-of-sample model. The 5-days out-of-sample data is the model that best fits the collected data and performs better to explain the observed prices, for the third loss function, according to table 7 that is provided below. The model that performs the worst when this metric is considered is the one that is based on 20 days of out-of-sample data.

RMSE Outputs from Part 3	
RMSE from BFM	0.2038
RMSE from OFS 1 Day	0.2021
RMSE from OFS 5 Days	0.1996
RMSE from OFS 20 Days	0.2133

Table 8 - RMSE Outputs from Heston Model Part III

This can be explained because the 5-days out-of-sample period is relatively short and may better capture the short-term dynamic of the financial market. In contrast, the 20-days out-of-sample period is longer and may introduce more uncertainty and variability in market conditions. Financial markets are subject to changes in volatility regimes, interest rates and other macroeconomic factors. Over a longer time horizon, these factor may change significantly leading to parameter drift in the Heston model, resulting in poorer performance.

Therefore, the 5-days out-of-sample model's superior performance in terms of % MSE for a short-term horizon can be attributed to is adaptability to the specific dynamics of the market within the timeframe. On that note, the 20-days out-of-sample, while potentially performing well in longer-term forecasting, may struggle to capture the nuances of shorter-term market movements. This can also be explained because the first DVF is simpler when compared to the other deterministic volatility functions which can impact on the out-of-sample analysis as well, meaning that the 20-days out-of-sample will cause in a bigger overfitting of the historical data.

For the paper presented by Christoffersen and Jacobs (2004) the best fitted model from the DVF is the one that has a better performance to explain the observed prices and the 5-days out-of-sample model is the one that performs worst.

Conclusions

As it was said before, the main purpose of this dissertation was to compare the Black-Scholes with the Heston model and understand which is the best model in terms of explaining the market dynamics. This dissertation was also focused on testing different specifications of deterministic volatility functions and evaluating the best loss function within each model. On that note, the objective was to assess whether a deterministic volatility function can be a good option valuation model, and after computing some tests, understand which of the purposed models best fits the observed data and better explains the volatility within the collected data.

In order to evaluate and determine if the deterministic volatility functions were a good approach for modeling the implied volatility and which loss function should be used when estimating and evaluating option valuation models, the dissertation followed the same path presented on both papers referred before (Dumas et al., 1998 and Christoffersen and Jacobs, 2004). Although the path was similar from the one registered on those papers, this dissertation didn't make the hedging and the prediction analysis made on the paper from Dumas et al. (1998).

Therefore, the focus was to analyze which of the models from the DVF would be better for each loss function, understand which model performs the best and also compare these values with the results computed on both papers. It is necessary to take into consideration that the data from the papers and the historical collected data presents a long interval between them and that can influence the results and, consequently, the conclusions.

The choice of the loss function is very important and can impact tremendously on the outputs. Regarding part 1 for the Black-Scholes model, the best model for the first loss function represents the BFM, which means that different values for the time to expiration can influence the results. For the Heston model, the 20-days out-of-sample model was the model that best fits the extracted data.

Taking part 2 into consideration, the best model for the second loss function is the model given by 20-days out-of-sample data. This means that a miss specified model with precisely estimated parameters (out-of-sample) outperformed the correctly specified model. For the Heston model, the BFM is the one that best explains the market dynamics.

Regarding part 3, the best model for the third loss function is the model 1 from the DVF, having, therefore, a better performance to explain the observed prices, regarding the implied volatility. The 5-days out-of-sample scenario is the model that best fits the collected data to the % MSE Loss function within the Heston model.

Based on the estimation results, it is possible to understand that the third DVF consistently performs well across different loss functions. However, the choice of the best model can vary depending on the specific context and objectives of the analysis implemented.

If we focus on all RMSE results for the two models, we can reach the conclusion that the IVMSE Loss Function is the best loss function within the usage of the Black-Scholes model and the % MSE is the best Loss function within the usage of the Heston model. But the purpose of this dissertation is to compare both models. On that note, looking at the estimation results from both models, we can conclude that if we focus on the IVMSE and the % MSE loss functions, the Heston model yields better results but if we focus on the $ MSE, the Heston model fits worse when compared to the Black-Scholes model. Therefore, it is possible to reach the conclusion that the Heston model is the one that best explains the historical collected data and better fits the market dynamics.

The paper from Christoffersen and Jacobs (2004) compares the Heston model with the PBS stating that "Regardless of whether one uses $RMSE or %RMSE loss functions, and regardless of whether one evaluates the loss functions in-sample or out-of-sample, the PBS model performs better than the Heston model when implemented using the appropriate loss function. However, when the PBS model is implemented using the IVMSE loss function to estimate the parameters, as is standard in the literature, it performs much worse than the Heston model." Therefore, the conclusion was that when comparing both models the PBS model slightly outperforms the Heston model.

Taking these conclusions into consideration, it is possible to understand that there is a discrepancy between conclusions. This can be explained for various reasons, such as market gaps, market dynamics and most importantly because of the collected data.

On that note, it is important to say that the results can be misleading considering that structural models like the Heston model can provide the key link between the dynamics of the option price and the dynamics of the underlying asset price. Therefore, the Blacks Scholes model only indicates its usefulness as a benchmark, and consequently it is not qualified to be a competitor of the structural models. There are also factors such as accuracy of calibration or market conditions that can impact tremendously on the results and consequently the conclusions. The historical data retrieved can also impact tremendously the outcomes and the conclusions.

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