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Public Stimulus to Private Investment and Prevention of Disinvestment: a CEV approach

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Masters in Finance

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I always loved everything that was music related and I still believe that music embodies something that will always be true to me – one can play a beautiful melody alone, but to me it always felt better to play accompanied by others, hearing everyone create beautiful harmonies, rhythm and dynamics. Writing this thesis was, to me, a lot like being a part of a musical ensemble. I could not have done it alone.

I have to thank each and everyone of my friends, who helped me to stay focused, but also to unwind when needed. A special thank you to those who offered their knowledge and helped me in the process of writing my thesis, be it with mathematical topics with which they were more comfortable with or on how to use LaTeX.

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Finally, not only during the period through which I was writing this thesis, but in fact throughout my whole life, my parents and my sister were always the ones guiding me along. Without their support I would not be where I am today, nor would my life be as full of beautiful moments as the ones it has been graced with.

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To all, thank you.

Resumo

Tomando em consideração o período atípico causado pelo COVID-19 e a guerra entre a Rússia e a Ucrânia, que criaram elevada incerteza económica, procuramos estudar como um governo limitado pela sua situação financeira poderá implementar uma política expansionista para estimular a economia. Como resposta, esta dissertação estuda a forma ótima do setor público conseguir estimular investimento privado imediato ou, não havendo essa possibilidade, prevenir o desinvestimento por parte do setor privado. Usámos uma abordagem baseada na literatura de Opções Reais e partimos do artigo de Barbosa et al. (2016), expandindo as suas ideias, que se focam na estimulação de investimento imediato sob um modelo com processo estocástico do tipo *Geometric Brownian Motion* (GBM), para tal incluindo o estudo da prevenção de desinvestimento e a derivação das expressões para ambos os cenários referidos num modelo de *Constant Elasticity of Variance* (CEV), mais complexo. Desta forma, pretendemos manter a consideração de fatores macroeconómicos relevantes, mas também melhorar o realismo e a precisão dos resultados obtidos. Adicionalmente, aplicamos os dois modelos a um caso específico, a partir do qual comparamos os nossos resultados com os obtidos por Barbosa et al. (2016) e o modelo GBM com o CEV. Concluímos que assumir um processo estocástico GBM quando o verdadeiro processo corresponde ao modelo CEV leva a uma subestimação do esforço a ser feito para estimular investimento imediato ou prevenir desinvestimento. Assim sendo, políticas de incentivo ao investimento que tenham por base um modelo GBM não irão ser capazes de produzir os resultados desejados.

Palavras-chave: Decisões de Investimento, Estímulo Público, Investimento Privado, Opções Reais, Modelo CEV

Classificação JEL: E22, H32

Abstract

Taking into consideration the atypical times caused by COVID-19 and the war between Russia and Ukraine, which have created great economic uncertainty, we sought to study how a government constrained by its financial situation could implement an expansionary policy to stimulate its economy. As an answer to this problem, this dissertation looks into the optimal way for the public sector to stimulate immediate private investment or, at the least, prevent disinvestment from the private sector. We use a Real Options approach and take the work of Barbosa et al. (2016), expanding upon their work, which focuses on the stimulation of immediate investment under a Geometric Brownian Motion (GBM) model, to include prevention of disinvestment and derive the expressions for both issues under a more complex Constant Elasticity of Variance (CEV) model. This way, we aim to keep the relevant macroeconomic factors considered but improve on the realism and precision of the results obtained. Moreover, we do an application of both models, through which we compare our results to the ones obtained by Barbosa et al, (2016) and compare the GBM to the CEV model. We conclude that assuming a GBM stochastic process when the true generation process corresponds to the CEV model leads to an underestimation of the effort to be done in order to stimulate immediate investment or prevent disinvestment and, hence, incentive policies put in place by the government with the GBM assumption will not reap the desired outcomes.

Keywords: Investment Decisions, Public Stimulus, Private Investment, Real Options, CEV Model

JEL Classification: E22, H32

Contents

CHAPTER 1

Introduction

In this dissertation we will study how the government can either stimulate immediate investment or prevent disinvestment from private firms, using a Real Options approach and, more specifically, a Constant Elasticity of Variance (CEV) model. This is a relevant topic, especially when in times of greater uncertainty, which has been the global scenario for the last couple of years. More than two years after its first appearance on Wuhan, China, COVID-19, an infectious disease which causes mild to moderate respiratory illness, is still a concern. Furthermore, in February of 2022, the war between Russia and Ukraine began, unfortunately leading to massive population movements and losses, and creating a massive global economic impact.

Regarding COVID-19, the "pandemic" classification of the outbreak was made on March 11, 2020. Although variants of the influenza virus have also been classified as a pandemic, even recently with H1N1 in 2009, the reality is that they never had the impact that COVID-19 had, both in number of infections and deaths and in how it overwhelmed the healthcare systems. This is why many look at the 1918 Influenza pandemic for some guidance on how this situation might evolve regarding Gross Domestic Product (GDP), consumption and other features of the economy, such as inflation, real returns on stocks or government bills (Barro et al., 2020, as an example). However, even if the two viruses share many similarities in clinical and pathological features, and in the public and government response, it is still difficult to compare events which are 100 years apart, given all the progress that has been made, with advances in technology, global supply chains, communication services and, especially, in the medical field, having a great importance.

When it comes to the war between Ukraine and Russia, it has created global tension, with many sanctions and other measures (removal of Russian banks from the SWIFT international payment system, for instance) imposed on Russia. Overall, the impact of this conflict is already having economic consequences around the world. The prices of energy (mainly electricity and gas) have increased dramatically creating pressure on businesses to increase their prices. Given this, the annual inflation rate has risen to 8.6% in the euro area in June 2022, and, specifically, to 9.0% in Portugal, according to Eurostat indicators[1]. The scarcity of cereals coming from a drop of supply from Russia has also created a food crisis, with high food prices creating insecurity and increasing hunger.

Taking all these considerations, it is no surprise that these phenomena have led to a global recession, with a decrease in annual GDP growth of 3.4% in the United States and

[1]Sourced from Eurostat. Accessed in 25/08/2022.

a decrease of 6% in the European Union in 2020 when the pandemic hit, as per the World Bank Data[2], and still negative or very low GDP growth rates in 2022, according to the Organization for Economic Co-operation and Development (OECD)[3], already accounting for the effects of war. On the one hand, China's response to the pandemic and the consequently decline in activity in the international markets created a disruption of supply chains, caused as well by the introduction of lockdown measures, temporary shutdown of some activities and social distancing rules around the world, which was only exacerbated by the war having a very similar effect in disrupting global supply; whilst, on the other hand, the demand was affected via rising unemployment, travel restrictions, lockdown and other government actions as well, and, after, by inflation leading to lower purchasing power.

It is expected that the effects of COVID-19 and the war will reverberate even after the outbreak is controlled and the war is over. This is because the economic decline that resulted from the pandemic was not only stemmed by government restrictions, but in large part caused by people voluntarily choosing to stay home and moving away from nonessential businesses, which may mean that the economic stimulus resulting from letting go of lockdown measures may be much smaller than expected (Goolsbee and Syverson, 2020). Similarly, the consequences of the war have been severe and the adjustment of companies and governments to the new economical and socio-political climate is expected to take some time.

Historically, crises put a strain on the levels of investment, with the impact connected to the level of tolerance to uncertainty (Inklaar and Yang, 2012), which is a huge factor when it comes to investment decisions (firms usually delay investment when under greater uncertainty). This phenomenon may also be explained by increased difficulty in accessing credit or lower levels of consumption, among many other possible factors, with the result being that firms end up cutting costs and reducing investment projects. This in turn feeds into the situation, as lower levels of investment mean lower employment in the future, decreased funding of research and development and, overall, lack of stimulus to the economy. This decreasing of investment levels has been observed as a consequence of past crises and as a result of the current issues. Specifically, looking at the Gross Fixed Capital Formation indicator, we see a negative quarterly growth rate for the European Union and the United States, for example, with value - 17.3% and - 7.2%, respectively, for the second quarter of 2020, right after COVID-19 was classified as a "pandemic"; and, even though still positive, these values for the first quarter of 2022, when the war began, are very low (0.2% and 1.2% respectively)[4].

Given this, Keynesian economics advise governments to implement expansionary policies, with emphasis on an increase in public spending and, more specifically, in public investment. The issue arises when we take into consideration financial constraints

[2]Sourced from World Bank Data. Accessed in 25/08/2022.
[3]Sourced from OECD Data. Accessed in 25/08/2022.
[4]Sourced from OECD Data. Accessed in 25/08/2022.

that governments may face, with the indebtedness level being one of them. In truth, public debt as a percentage of GDP increased drastically in 2020 as a result of the disruption caused by COVID-19, with Japan, Greece, Italy, United States and Portugal occupying the top 5 spots in this indicator (respectively, 257%, 238%, 184%, 161%, 157% as the value of public debt as a percentage of GDP). These values were still high in 2021 (values for Japan, Greece, United States, Portugal and United Kingdom are 259%, 222%, 150%, 145% and 143%, respectively)[5]. Such levels of indebtedness have a negative impact on the long-term growth (Checherita-Westphal and Rother, 2012) and so the implementation of a public investment plan and its financing through an increase in debt may not be ideal. Already in the response to the financial crisis of 2008 we saw countries in situations similar to these have to resort to austerity measures in order to maintain a grip on their financial situation, with a public investment plan not being the priority at all. This is the most likely scenario when countries are in a situation of crisis and face indebtedness levels such as these. Hence, governments must find an alternative in order to ensure that the stimulus to the economy still exists, which may be possible with turning to private investment. Promoting immediate private investment or preventing disinvestment instead may involve lower financial strain on the public sector and still reap the desired result of stimulating the economy.

In an attempt to study the alternatives available to the government in order to achieve what has been proposed, we take the work of Barbosa et al. (2016), which studies precisely the dynamics between public and private sector and attempts to derive the optimal behavior regarding investment opportunities, via a Real Options approach. The authors have created a very complete model in which they take into account investment multipliers, two types of taxes (economy wide tax and corporate tax) and public inefficiency, and which allowed them to conclude on the different policies that can be used by governments to stimulate private investment regarding not only the optimal way to set them up, but also when would instead be preferable that the public sector take the project itself.

Seeing that this work lays out the foundations for what we aim to discuss within this dissertation, we decided to replicate it and expand upon it. What this dissertation offers is a more realistic view into the issue at hand, via modifications on the model used by Barbosa et al. (2016). Being so, this dissertation not only updates the results obtained by Barbosa et al. (2016), but also compares them to the ones obtained when using a model which introduces a non-constant variance and is mathematically more complex – the CEV model. Furthermore, it introduces the topic of prevention of disinvestment, as a way to at least maintain investment in projects which are relevant to the country's economy. Via using the more complex CEV model, we aim to study if the increased complexity is rewarded with more insightful results. We theorize that not only this is so, but also that the structure of the CEV model allows for a more detailed study of the

[5]Sourced from OECD Data. Accessed in 25/08/2022. Data for Italy for 2021 was not available at the time of consultation.

issue at hand via incorporation of characteristics of different projects under consideration that can be reflected into the "CEV-exponent". That is, this exponent may be a way to increase the precision of the results arrived to, when its value is calibrated to the specific investment opportunity that is being studied. One particularly interesting point is that the model of Barbosa et al. (2016) is actually a particular case of the CEV model, when the CEV-exponent is equal to 2. This is relevant to our work, as it facilitates the comparison between the two models. Incidentally, we not only perform this analysis but also recreate the application that Barbosa et al. (2016) present in their work regarding the Portuguese economy.

From the derivations and numerical analysis presented henceforth, we are able to achieve the objectives that we have laid out, being able to derive the expressions under both the GBM and the CEV model that are useful for the government when creating its incentive strategy for investment. We have derived these expression for the issues of stimulation of immediate investment and prevention of disinvestment, as we have aimed to do. Moreover, our numerical results, especially in their comparison between the GBM and CEV models, have provided the insights we were hoping for. In truth, they have shown that if the government wrongly assumes a GBM process for the stochastic variable when its true generation process is one translated into a CEV model, the incentive policies put in place would not achieve the desired results and neither stimulation of immediate investment nor prevention of disinvestment would ensue. Our application of the models that we have derived show that the effort that is needed from the government is greater if a CEV model is considered, meaning that the threshold for investment is higher, and the threshold for disinvestment is lower.

Taking all that has been said, the work presented herein is structured as follows. Chapter 2 will be a review of the literature regarding the topic at hand – public stimulus of private investment and prevention of disinvestment – and of the CEV model, whilst Chapter 3 will be a review of works pertaining to the relevant macroeconomic factors that will be taken into consideration. In Chapters 4 and 5 we derive and present the expressions of the Barbosa et al. (2016) model and of the CEV model, respectively. Chapter 6 is where we produce and analyze an application to the Portuguese economy. Finally, in Chapter 7 we conclude.

CHAPTER 2

Literature review

The paper from which this dissertation draws inspiration is Barbosa et al. (2016). Using a Real Options valuation approach rooted in a Geometric Brownian Motion (GBM) process that governs the pre-tax profit flows of the project, the article explores various alternatives that the government can use to stimulate immediate private investment. In particular, the authors consider the possibilities of an investment subsidy and a corporate tax whilst also weighing the consequences of the government acting as a competitor to the private firm for the project. The paper innovates from previous research because it introduces two types of taxes (profit tax rate and average income tax rate for the economy), investment multipliers, and consideration of public inefficiencies. This paper shows, among other conclusions, that a subsidy policy is preferable to a tax reduction given the variables and considerations taken into account in the modelling presented. Moreover, it demonstrates that the optimal subsidy can be reduced if the government acts as a competitor to the private firm. In this dissertation, we expect to reach similar conclusions. Our hypothesis resides within the values of optimal subsidy and tax reduction when we consider a more realistic setting, which we will introduce by modifying the framework used by Barbosa et al. (2016). We believe that these values will be affected by the new model we will be using. The comparison between the values we will obtain and those obtained by Barbosa et al. (2016) is straightforward, given that we replicate this paper whilst also computing the values under our modified model. Additionally, we will also study the issue of investment during uncertain times through the opposite lenses. This is because we will look into stimulation of immediate investment and also into the prevention of disinvestment. We do this in order to include instances in which the government wishes to not lose investment in some projects which are relevant to the economy.

The Real Options approach has been the preferred one to study the issues at hand, since it incorporates uncertainty in the calculations performed, with the foundations of its literature coming from Black and Scholes (1973) and Merton (1973). McDonald and Siegel (1986) emphasize the importance of uncertainty by demonstrating the existence of an optimal investment threshold that determines the timing for exercising the option to invest and showing that such a value can be much higher than that suggested by the better known Net Present Value (NPV) rule, as it incorporates the value of waiting that arises from uncertainty. This topic is also the focus of Dixit and Pindyck (1994), where the authors delineate the basic assumptions and methodology of Real Options for the case where the stochastic variable follows a GBM.

For the issue of prevention of disinvestment, the literature surrounding this topic is quite scarce, being more focused in specific cases. See, for instance, Lambrecht and Myers (2007), who use a Real Options approach to study takeovers and disinvestment in firms that operate in declining industries, or Chambers et al. (2017) who instead perform an empirical analysis of the reduction of investment in inefficient or low-value health care services within health care systems. Both examples provided highlight how the literature involving disinvestment, besides very little, is very specialized in the topic it covers. See that Lambrecht and Myers (2007) focus on takeovers as the process of disinvestment under analysis, whilst Chambers et al. (2017) focus on disinvestment in health care. What is more usual, nevertheless, is the study of entry and subsequent exit options, which means that we are considering an option to invest and disinvest later if the conditions deteriorate. This is not, however, what we are aiming for in our work, since we assume that the firm is already in the operating state, and so the only option available is the one for disinvestment. Given this, we see that our contribution may prove to be very beneficial to the discussion, as it provides a new outlook on disinvestment and how the government can prevent it from happening when uncertainty increases.

When it comes to the stimulation of investment, however, the literature is vast and worth exploring in more detail. As mentioned, other works have relied on Real Options to study public incentives for private investment. In this respect, the work of Pennings (2000) has some similarities with Barbosa et al. (2016), as it also uses a Real Options model to study the impact of introducing an investment subsidy and a tax on future profits on the investment trigger, with the pre-tax profit flows being the stochastic variable (following a GBM as well). This paper shows that a reduction in the critical threshold is possible at an expected cost of zero to the government. Furthermore, Pennings (2000) shows that the tax rate required to guarantee the zero expected cost to the public sector decreases when the group of investors is more heterogeneous. Considering the heterogeneity of the group of investors is quite relevant because, when the government sets public policy, it does so for all possible firms that are interested in the investment opportunity, which most likely have different profiles regarding their exposure and aversion to uncertainty. In another one of his works, Pennings (2005) revisits this topic and shows that using a Real Options approach and assuming a two-stage game where the government sets a policy and the firm then decides when to invest, under no cooperation, the best alternative is to maximize both the subsidy granted and the tax levied. However, such a policy would be difficult to implement because there are usually limits to subsidies granted.

When making a concrete comparison between subsidy and tax cut as the better policy to encourage investment, some authors conclude that subsidy is the better alternative. Like Barbosa et al. (2016), Yu et al. (2007) concur that investment subsidies are more efficient and economical and thus dominate tax rate reductions. Via the propositions presented in their work, they demonstrate that, in comparison with tax reduction policies, entry subsidies result in a lower present cost when investment trigger values are held equal for both policies, and in a lower threshold value when the present cost is the same for both subsidy and tax reduction. Nevertheless, when considering changes in the tax policy, it may also be important to reflect on an interesting result presented by Hasset and Metcalf (1999), which has to do with investment and uncertainty regarding the tax policy. The authors show that higher uncertainty regarding tax rates and their future evolution may increase investment, as investors are worried that tax rates may increase even more – thus, they prefer to invest at the now relatively lower rate. Therefore, tax reduction may help increase the uncertainty regarding their evolution and produce this effect as well, which may be an advantage when compared to investment subsidies. However, Morisset and Pirnia (2000) present a review on tax policy and its influence over Foreign Direct Investment (FDI) and actually show that tax policy has a limited effect, being that FDI is more greatly affected by the countries' political and economic stability and infrastructure, among other similar factors. Hence, in considering investment and its stimulus, the overall characteristics of the country must also be taken into consideration when deciding on which is the better policy.

Also relevant to this debate is the work of Sarkar (2012), where the author focuses on the impact of discount rates used by the government and the private firm, specifically on the implications of considering them equal or different, concluding that the assumption regarding this is significant in determining whether a subsidy or a tax policy is better. Namely, when one assumes that both discount rates are constant over time, if they are equal then subsidy dominates tax reduction; but, when otherwise, the policy implemented should rely entirely on tax reduction or maximize both investment subsidy and tax rate. If the discount rate of the government increases with the amount of subsidy, then it may be optimal to choose a combination of both an investment subsidy and a tax reduction. In truth, the combination of both policies is quite common, even though it may seem that the government is taking with one hand and giving with the other, which is explained by Hansson and Stuart (1989) as the consequence of sequential governments being the ones deciding on the policy to implement. Also, and referencing again Morisset and Pirnia (2000), the authors show that developing countries often opt for the combination of tax policies and investment incentives. These, however, have not proven to be successful given the overall unattractive investment climate in those areas. Danielova and Sarkar (2011) show instead that a mix of investment subsidy and tax cut policy is optimal when considering the possibility of debt financing by the firm. This result may come from the fact that a high tax rate, even though bringing more revenue to the government, also leads to more indebtment from the firm, increasing the bankruptcy risk and the investment trigger value. The overall result is the need for a higher investment subsidy to induce immediate investment.

One interesting recent study, which does not rely on Real Options theory but rather on a natural experiment in Poland, also provides relevant insights into the impact of either policy when in different economic conditions. Guceri and Albinowski (2021) analyze two very similar subsidies put in place two years apart – one in a period of stability and the other in a period of high uncertainty. They conclude that even though tax incentives seem to have very positive results on investment in periods of high economic stability, under high uncertainty a large share of the firms prefer to "wait and see", even when the incentives are substantial. In such times, the impact of the subsidy or tax policy is largely dependent on the degree of exposure of firms to elevated uncertainty.

Moving on to more technical aspects of the work presented herein, and after having looked into the conclusions of other works which analyze the issues to be discussed, it is just as important to dive deeper into the methodology being used. Even though we concur that the Real Options framework, as used and modeled by Barbosa et al. (2016) and many of the papers referenced, is well suited to the problems at hand, we believe that there are ways to improve upon it, being that this is the main driver of the work to be presented in this dissertation.

In order to dive into the reasoning behind why we make such a statement, it is paramount that we explain exactly which are the issues that we identify in the model used by Barbosa et al. (2016). Barbosa et al. (2016) and, as stated, most of the papers already referenced in this Literature Review, use a GBM to model the stochastic variable (probably mainly because of its easier analytic tractability). In doing so, the authors assume that the implied risk-neutral probability densities follow a lognormal distribution. It is important to remember that, compared with the more well-known normal distribution, since the values in a lognormal distribution are always positive, the resulting curve is not symmetric but rather right-skewed. Nevertheless, it has been found that risk-neutral probability densities are actually heavily skewed to the left and leptokurtic (Jackwerth and Rubinstein, 2012). When a distribution is leptokurtic, and once again comparing with the normal distribution, this means that there are more observations in the middle and on the tails than the normal, which in turn means that bigger variations are more likely to occur than if a normal distribution is assumed. Hence, taking the empirical findings of Jackwerth and Rubinstein (2012) regarding the skew and kurtosis of the risk-neutral probability densities, we already detect that assuming that the stochastic variable follows a GBM may lead to misleading results, as also pointed out by Lund (1993) who similarly argues that the GBM is not a plausible equilibrium price process.

Furthermore, in choosing such a stochastic process, the authors also assume no volatility smile. The volatility smile is the term used when implied volatility is greater for deep out-of-the-money and deep in-the-money options, meaning that the implied volatility is a convex function of the strike price. Moreover, using a GBM results in an inability to capture time-varying drifts or time-varying volatility, with the constant value assumption for both parameters leading to the expectation that the stochastic variable will grow exponentially over time (Kanniainen, 2009). There is also the impossibility of accounting for mean reversion, a phenomenon that has been shown to exist by Gibson and Schwartz (1990) and Laughton and Jacoby (1993), for example.

Regarding mean reversion, however, it must be made clear that it will also not be taken into account in this dissertation. It has in fact been shown that mean reversion exists, as stated before, and particularly, when talking about commodity prices, it is expected, as the result of supply and demand forces. That is, when prices rise, either firms expand production or new ones enter the market, resulting in a supply increase, which in turn dampens the price increase. During this process of reaching equilibrium, prices may fall, leading to firms decreasing investment or exiting the market. Furthermore, there may also be demand shifts (because of change of preferences or increase in customer base, for whatever reason). Mean reversion can be a way of taking into account these supply and demand dynamics, which via affecting the product price affect the investment value of a project that may be being taken into consideration by a private firm. But, regarding the addressing of mean reversion in Real Options analysis, there are those who argue that the GBM is incapable of capturing it (Bessembinder et al., 1995, Schwartz, 1997, e.g.) and those who state that using a GBM can still be a reasonable assumption in models of investment under uncertainty. Metcalf and Hasset (1995) explain that two effects offset each other when comparing a GBM process and a GMR (Geometric Mean Reversion), using the latter as an example of a way to model mean reversion. These two effects are the "variance effect" (accounting for mean reversion decreases the long-run variance and, therefore, the uncertainty, leading to higher investment) and the "realized price effect" (not accounting for mean reversion and considering an increased variance level means that higher prices can be achieved, inducing higher investment). The argument made is that these two offset each other in such a way that expected cumulative investment is the same under GBM and GMR. In our case, taking this argument as plausible and considering that we will already be dealing with the CEV model, which increases the level of complexity by assuming a non-constant variance, we decided not to include mean-reverting properties in the stochastic process, in order not to overcomplicate the model presented. What we will address, however, are all the other issues pointed out. To do so, as has been said, we will use a CEV model.

Cox and Ross (1976) had already argued that the underlying stochastic process that determines the movement of the stock is a key factor in the option valuation, and, in this paper, they introduce alternative jump and diffusion processes. Nevertheless, and as stated, we will focus on the CEV model, which was proposed by Cox in an unpublished note in 1975 (Cox, 1975), and later published formally on Cox (1996). The main change introduced was in the way volatility was modelled, no longer constant, but rather conditional on the asset price level. Being this way, it is possible to incorporate the implied volatility smile which is ignored when considering a GBM process.

One important component of the variance function in the CEV model regarding this point is the "CEV-exponent", β. For different values of β, the model can capture the different types of volatility skew. For $\beta < 2$ the model incorporates a negative volatility skew, which is mostly connected with stock index options and crude oil prices; for $\beta > 2$ a positive volatility skew, characteristic of commodity spot prices; for $\beta = 2$ we have the specific case of the GBM (Geman and Shih, 2009). The value of β is obtained from historical data, when possible - for instance, Geman and Shih (2009) use the Generalized Method of Moments, as defined by Hansen (1982) and Chan et al. (1992). Besides this, the CEV model is furthermore able to capture the leverage effect, which refers to the negative relation between stock returns and their volatility (when the value of the stock decreases, financial leverage, which is the ratio between total debt and shareholders' equity, increases, making the stock riskier and, therefore, increasing its volatility). This effect has been documented by, for example, Bekaert and Wu (2000), and it is also captured by the CEV model via the volatility being a function of the asset price.

Cox (1996), which we already referred to previously, derives the closed-form solutions for European-style options for when $\beta < 2$. Similarly, Emanuel and MacBeth (1982) derive these expressions, but specifically for when $\beta > 2$, and show that the CEV model is able to obtain better predictions for future option prices than the Black-Scholes model. Nevertheless, we will focus on the works of Davydov and Linetsky (2001) and Davydov and Linetsky (2003), which also look into European-style contingent claims, and, with greater emphasis, on Dias and Nunes (2011), where the authors derive analytic solutions for perpetual American options.

Going back to the points made regarding the "CEV-exponent", we would like to clarify that we will focus on its value during our analysis. We hypothesize that the value of optimal subsidy or tax reduction needed to stimulate immediate private investment or the tax values which prevent disinvestment will be affected by the β value. When specifically talking about the stimulation of immediate private investment, and taking the results of Dias and Nunes (2011) we see that the value of β can increase or decrease the investment trigger, depending on the relation between the investment yield and the risk-free rate. Being so, we believe that when the investment trigger value is decreased, when compared to the $\beta = 2$ (GBM) model, then the value of optimal subsidy will be lower than if the model of Barbosa et al. (2016) is assumed; and will be higher when the investment trigger is increased. The same rationale applies for tax reduction, which will be lower (higher) when the investment trigger value is decreased (increased).

CHAPTER 3

Relevant macroeconomic factors

We will now explore in greater depth the aspects that constitute the innovation introduced by Barbosa et al. (2016) and that we will consider both in the replication of their paper and in our CEV model. These relevant macroeconomic factors must be looked upon critically since, even though not related to Real Options theory, they are an essential part of the framework used throughout this dissertation. They are explored in Barbosa et al. (2016) and will also be so in the following subsections.

3.1. Investment subsidy and taxes considered

Working within the Real Options framework, we know that, assuming positive NPV, the way to accelerate investment is by eliminating the value of waiting. Armada et al. (2012) concur, as they conclude that the cost associated with the incentive must equal the value of the option to defer, which the firms lose when investing immediately. Similarly, disinvestment happens when the value of waiting for better conditions decreases and the better decision is to exit the project. In our work, subsidy and tax policies will be the instruments we will use when it comes to stimulation of investment, and only tax policy will be considered when it comes to prevention of disinvestment.

Subsidy and tax reduction policies are different in their form but also in the way they impact the investment decision. During the work presented in this dissertation, and as per Barbosa et al. (2016), the subsidy under consideration will be an investment subsidy, hence why we do not consider it a viable option when it comes to preventing disinvestment. As studied in Armada et al. (2012), who consider and compare various subsidies and other policy options (investment subsidy, revenue subsidy, minimum demand guarantee and rescue option), the authors stress that the cashflows of an investment subsidy only occur at the moment of investment. As they also explain, if, instead, the government were to opt for a revenue subsidy (which is the one that increases the value of the investment opportunity the most in their results), then this would show a higher future commitment to the company. Nevertheless, considering an investment subsidy is very common in the literature regarding the stimulus of private investment[6]. Notice then that while the subsidy chosen - an investment subsidy - immediately shifts part of the sunk cost from the firm to the government, the tax (reduction) is more related to a sharing of uncertainty on the future profits (Pennings, 2005), which explains why both policies are different in the way they may impact the decision to invest.

Regarding taxation, along with the also commonly used profit tax[6], Barbosa et al. (2016), and therefore us as well, consider an income average tax rate on the economy, which is paramount to capture increased or decreased government revenue – these changes occur because of multiplier effects which will be considered too and further explained in the following subsection.

3.2. Investment multipliers

The impact of public investment on private investment is important to consider and is usually one of two: a crowding-in or a crowding-out effect. On the one hand, crowding-in occurs if an increase in public capital encourages private investment, which can happen mainly because of the possible complementarity between both types of investment – meaning, better public infrastructure can reduce costs and increase the productivity of private capital. On the other hand, crowding-out may transpire if there is competition between government and private sector for the same resources, or if investment in public capital is subsidized by public debt, which, via an increase in public deficits, creates a reduction in available credit and/or an increase in interest rates.

The literature regarding the debate on which of the two effects dominates is vast, with the discussion producing different results and conclusions regarding the matter. On one side, some claim that crowding-in is the stronger effect. Aschauer (1989) defends that this is the case, arguing that the increase in public capital stock raises the marginal productivity of private capital and, thus, private investment. However, the results that Aschauer (1989) presents, including the high output elasticity with respect to public capital estimated, were afterwards questioned on econometric grounds. Nevertheless, other authors agree that crowding-in may be the dominant effect, namely Seitz (1992), Argimon et al. (1997), and Erden and Holcombe (2006). On the other side of the debate are those who argue for crowding-out, Voss (2002) being one example.

One important argument in this discussion is that the type of investment and the overall economic conditions and country specificities may be key factors in what effect dominates. Pereira (2001) arrives at results that support this view since the author concludes that public investment crowds-out private investment at the aggregate level, but that crowding-in is strongly recorded for the specific cases of industrial and transportation equipment, and marginally for structures. In the same vein of thought, Perotti (2004) clearly presents the argument for the importance of the type of investment and country conditions, remarking that the same investment project in one country may have different marginal productivity in another. This may be true for countries with different levels of GDP or public capital per capita, being that the higher value these two indicators have, the more probable it is that the marginal product of a public investment project will be close to zero. Therefore, Perotti (2004) advises that the capacity of public investment as a stimulant of economic growth should be doubted, an opinion shared also by Romp and de Haan (2007), for the same motives.

[6]Examples of both an investment subsidy and a profit tax used as instruments for private investment stimulus are present in Hansson and Stuart (1989), Pennings (2000), Pennings (2005), Yu et al. (2007), Danielova and Sarkar (2011), and Sarkar (2012).

This idea that countries with lower GDP or lesser public capital per capita could be more prone to show a crowding-in effect of public investment is somewhat addressed by Cavallo and Daude (2011), where they study crowding-in versus crowding-out in developing countries. On one side, we see in such countries the potential of public capital having a stronger positive effect on private investment as explained by Perotti (2004); however, weak institutions and poor functioning financial markets that create added difficulties in the access to credit actually dampen this potential so much that crowding-out ends up being the prevailing force. The conclusion that the country's environment is the main responsible for this result is corroborated by the fact that Cavallo and Daude (2011) also show that for countries with better institutions the crowding-out effect is not as strong.

Overall, it then seems that the quality of public investment, rather than its quantity, is decisive to the effect it creates on the economy. Therefore, and as agreed by Barbosa et al. (2016), it is not wise to spend resources on lower-quality public investment projects or unproductive ones, as that could lead to a crowding-out effect and, thus, a negative impact on the economy.

One way that the effect of public investment or private investment on the economy can be measured and taken into consideration is through investment multipliers. Essentially, the investment multiplier tells us how much a given increase in investment increases effective demand as a whole. The value of such indicators can be obtained, for instance, via a VAR (Vector Autoregression) approach, which is what Afonso and St. Aubyn (2010) use in their work. The authors compute the rates of return for public and private investment and conclude that, based on the values obtained, private investment not only always has a positive effect on output, but that its effect is always greater than if public investment is considered.

This way, taking what has been discussed in this section, it logically follows that considering the investment multipliers is relevant to our work, seeing that it may be a critical factor in deciding whether the private firm or the government should go ahead with the investment opportunity. Hence, just like Barbosa et al. (2016) introduce in their model private and public investment multipliers, we will do the same in replicating their model and in our CEV approach.

3.3. Public sector inefficiencies

Public sector inefficiencies can arise from poor selection and implementation of projects, which may result from weak institutions and/or corruption, related to wasted resources and lower technical knowledge. The inefficiencies usually lead to higher costs to the project and lower returns.

As was stated in the previous section regarding investment multipliers, the quality of public investment is decisive when it comes to its impact on the economy and private investment, and quality is related to the level of efficiency that a government can withhold in its investment projects. However, many authors have identified flaws in public sector practices that lead to the inefficiencies referred.

Rajkumar and Swaroop (2008) argue that public investment in some sectors, such as health and education, can be an easier option than actually addressing the governance issues that lead to inefficiencies, but also show that simply increasing public spending in those areas does not guarantee an improvement of the country's institutions. At the same time, some authors focus on what specific factors may create efficiency issues, in an attempt to not only identify them but also find ways to overcome them. Chakraborty and Dabla-Norris (2009) specifically home in on corruption and argue that government oversight is less effective with a greater level of corruption. Given that screening, monitoring and oversight may be significant to control investment returns, if these are not successfully in place, then the quality of the project itself will suffer. The conclusion reached by Chakraborty and Dabla-Norris (2009) is mainly related to institutions – the weaker the institutions are in any one country (usually connected to a greater level of corruption), the less efficient public investment will be and, therefore, increasing its level may not lead to the desired effect of economic stimulus, but rather to a negative impact on the overall economy. Leeper et al. (2010) concur with this conclusion, but their work focuses instead on inefficiencies caused by implementation delays and the consequent need for adjustments in fiscal policy, to accommodate extra costs, for instance. In truth, the authors contend, delays increase the time needed for the public capital to increase the marginal productivity of private capital; hence, private investment is postponed until public capital is able to do so.

Afonso et al. (2005) and Dabla-Norris et al. (2012) attempt to measure the degree of public (in)efficiency, with the former computing public sector performance and public sector efficiency indicators via consideration of multiple socio-economic factors for 23 industrialized countries, and the latter constructing an index of public investment management efficiency, via aggregation of indicators across four key stages of the investment process (strategic guidance and project appraisal; project selection; project management and implementation; and project evaluation and audit), doing so for 71 countries, including 40 low-income ones. Notice how the approach used in both papers is similar, with the methodology relying on several factors that overall determine the efficiency of a country's public sector. That is, it shows how much of a complex subject it is, entailing several facets of a country's government, institutions, economy and society.

Once again, taking the points that were made, it is essential that, like Barbosa et al. (2016) have done in their work, we consider public inefficiencies, accounting for them in the investment costs (which are higher if inefficiency is greater) and in investment profits (which are lower when there is high inefficiency).

CHAPTER 4

Methodology - GBM model

As already stated, the work developed in this thesis follows and then extends the analysis of Barbosa et al. (2016) to other models that are better able to accommodate some empirical regularities often observed in the markets. This section of the dissertation will contain firstly the explanation of the base article framework and, afterwards, the development of the alternative models and enhancements built on it.

4.1. Base model - GBM

In Barbosa et al. (2016), the authors assume the existence of a project (or set of projects) important for the economy, which can be implemented by the private or public sector. The aim, as explained before, is to compare the alternatives: (i) government being the one to go through with the project (which is the case when the investment trigger has not been reached for the private sector); (ii) the private company being the one to invest, without any need for government intervention; and (iii) the government modifying the project (through subsidy or taxation) to make it more attractive and stimulate immediate private investment. In our work, as stated, we will also look into prevention of disinvestment, using the same base modeling, but adapted to that scenario. In this case, we aim to study if the private firm is induced to exit a project which, once again, is considered important to the economy, and what the government can do to prevent it from doing so.

In the setting used in this paper, all variables are considered constant (given reasonable values), with exception of the value of the pre-tax profit flows (V), which follows a GBM (this is the only source of uncertainty in this setting):

$$dV = \alpha V dt + \sigma V dz, \tag{4.1}$$

where α is the expected profit flows drift and σ is the instantaneous volatility (both assumed to be constant), and dz is the increment of the Wiener process. Additionally, $\alpha < r$, where r is the risk-free interest rate, and all entities are risk neutral (meaning we are already working with an equivalent martingale measure).

Note that $\alpha < r$ guarantees that the dividend yield of this project is positive (i.e., $q > 0$), as otherwise the model collapses. It is only the presence of a non-zero cost of waiting that makes problems with infinite horizon reasonable, because if waiting were costless, perpetual call or put options, as the ones we are considering here (decision to invest or to disinvest, respectively), would never be exercised.

Following the notation of the paper. $H(V)$, with $H = \{P, G\}$, represents the value of the investment/disinvestment opportunity for the private firm ($H = P$) or the value of the investment opportunity to the government ($H = G$), since either can implement the project, but we only consider the possibility of the private firm disinvesting in a relevant project. Hence, and using this notation, we follow the standard real options valuation theory, according to which the investment/disinvestment opportunity $H(V)$ must satisfy the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2 V^2 H''(V) + \alpha V H'(V) - rH(V) = 0. \tag{4.2}$$

This ODE, as is known, is related with the partial differential equation (PDE) from the Black and Scholes and Merton model (Black and Scholes, 1973; Merton, 1973), with the caveat that for infinite horizon options, such as the one we are working with, the problem becomes time independent, which then leads to the ODE (4.2). Its general solution is:

$$H(V) = A_H V^{\beta_1} + B_H V^{\beta_2}. \tag{4.3}$$

This is the base of our model when having a GBM for the stochastic process. Now we will adapt this methodology to the two issues we wish to study: stimulation of investment and prevention of disinvestment.

4.2. Stimulation of investment - GBM

Since we are considering an investment opportunity (i.e., a call option), the following boundary conditions must be met:

Boundary Condition I: $H(0) = 0$;
Boundary Condition II: Value-Matching Condition;
Boundary Condition III: Smooth Pasting Condition.

Boundary condition I comes from the fact that if the project's value is zero, then the option to invest will be of no value, with its owner preferring to remain in the idle state rather than acting upon a worthless option to invest. Boundary condition II, or value-matching condition, essentially states that by investing in the project the firm receives a net payoff equal to the value of the project minus the direct cost of investment. Finally, boundary condition III, or smooth-pasting condition (also known as high-contact condition), ensures that $H(V)$ is in fact continuous and smooth at the critical exercise point or investment threshold (represented as $\overline{V}_H$), making it in fact the optimal point for investment. Both the value-matching condition and the smooth-pasting condition are to be set in accordance with if the private firm or the government are considering the investment decision.

To accommodate Boundary Condition I, it is necessary that $B_H = 0$, which means that the solution for our specific case becomes:

$$H(V) = A_H V^{\beta_1}. \tag{4.4}$$

A_H is a constant to be determined from the other two boundary conditions (along with $\overline{V}_H$), being it dependent on specific payoffs for the private firm or the government. β_1 is the positive root of the fundamental quadratic equation $Q(\beta) = 0.5\sigma^2\beta(\beta - 1) + \alpha\beta - r = 0$ (which comes from the process of solving the ODE mentioned previously), being equal to:

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \tag{4.5}$$

The complete derivation of equations (4.1) to (4.5) can be found in Appendix A, in which we have relied on Shreve (2004) to arrive to the expressions aforementioned.

4.2.1. The private firm undertakes the project

We first must analyze the investment decision for both the private firm and the government, without any policy in place. We start with the private firm ($H = P$), with the important consideration that the firm pays corporate tax over the profit flows (we recall that V represents the pre-tax profit flows). We then begin by setting boundary conditions II and III, in accordance to this:

Value-Matching Condition: $P(\overline{V}_P) = \overline{V}_P(1 - t_c) - \overline{X}_P$
Smooth Pasting Condition: $P'\left(\overline{V}_P\right) = 1 - t_c$,

where the subscript P in $\overline{V}_P$ and $\overline{X}_P$ denotes that these are the investment trigger and investment cost for the private firm, respectively, while t_c is the capital income tax rate. Both the conditions are in accordance to their usual formulation, being the value-matching condition the payoff of investing, which for the firm corresponds as is stated to the after-tax profit flows minus the direct cost of investment, and the smooth-pasting condition, which is the derivative of the value-matching condition with respect to the investment trigger $\overline{V}_P$.

Given these boundary conditions, the solution can be found (as shown in Appendix B) and one can compute the value of the investment opportunity, as well as that of the investment trigger:

$$P(V) = \begin{cases} \left(\overline{V}_P(1 - t_c) - \overline{X}_P\right)\left(\frac{V}{\overline{V}_P}\right)^{\beta_1} \text{ for } V < \overline{V}_P \\ V(1 - t_c) - \overline{X}_P \text{ for } V \geq \overline{V}_P \end{cases} \tag{4.6}$$

and

$$\overline{V}_P = \frac{\beta_1}{\beta_1 - 1}\frac{\overline{X}_P}{1 - t_c}. \tag{4.7}$$

Even if the private firm is the one taking the project, the government's payoff must still be taken into account, since it will gain not only the corporate tax value over the project's profit flows, but also an increase in taxes collected in the overall economy since, as explained, the investment made by the private firm will lead to a change in economic activity, measured via its multiplier.

This way, the public sector will have the following payoff:

$$\Pi_G = t_c V + t_n \lambda_P \overline{X}_P, \tag{4.8}$$

where t_n is the normal average tax rate for the economy and λ_P is the private investment multiplier.

Note, however, that if $V < \overline{V}_P$, then the private firm will not make an immediate investment. Therefore, the public sector can undertake the project itself, which is a possibility that will be studied as well, but that, as shown in the Introduction of this dissertation, may prove to be difficult in some cases because of fiscal or debt constraints, among others, or have a lower impact in the economy, given the inefficiency of the public sector, also explored before. Hence, the government may prefer to change the investment conditions in order to stimulate immediate private investment in the project.

4.2.2. The public sector undertakes the project

As mentioned, the public sector can opt to undertake the project, instead of stimulating immediate private investment. We again begin by setting the boundary conditions II and III:

Value-Matching Condition: $G\left(\overline{V}_G\right) = \overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G$
Smooth-Pasting Condition: $G'\left(\overline{V}_G\right) = 1.$

In this case, $\overline{V}_G$ and $\overline{X}_G$ are, respectively, the investment trigger for the government and its direct investment cost, whereas λ_G represents the investment multiplier for the public sector. As before, t_n is the normal average tax rate for the economy. Note that the value-matching condition once again ensures that, when investing, the owner of the option (the government in this instance) receives the payoff of investment minus its direct cost. In this case, the payoff is more complex, because (i) the capital income tax rate has a neutral effect for the government, and (ii) the government will receive the change in tax collected resultant from the change in economic activity caused by the investment.

In Barbosa et al. (2016) the authors also account for inefficiencies from the public sector, both regarding the public investment cost (relative to the private investment cost) and the current value of the pre-tax profit flows for a government running the project:

$$\overline{X}_G = (1 + \gamma_X)\, \overline{X}_P \tag{4.9}$$

and

$$V_g = (1 - \gamma_V)\, V, \tag{4.10}$$

where γ_X is the rate of public inefficiency for managing the implementation of the project (leading to a possible higher direct cost of investment) and γ_V is the rate of inefficiency for extracting the profits.

Given these considerations, the values of the investment opportunity and of the trigger to invest are (as demonstrated in Appendix C):

$$G(V) = \begin{cases} \left(\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G\right)\left(\frac{V_g}{\overline{V}_G}\right)^{\beta_1} & \text{for } V_g < \overline{V}_G \\ V_g - \overline{X}_G + t_n \lambda_G \overline{X}_G \quad \text{for } V_g \geq \overline{V}_G \end{cases} \tag{4.11}$$

and

$$\overline{V}_G = \frac{\beta_1}{\beta_1 - 1}(1 - t_n \lambda_G)\overline{X}_G. \tag{4.12}$$

4.2.3. The government subsidizes the project and the optimal incentive policy

The government may decide to change the project in order to stimulate immediate private investment. One way to achieve this is through a subsidy policy. The model will be derived in the same way as when we were considering the case in which the private firm invests, with the difference that now the direct cost of investment is diminished by the subsidy value, which will be denoted by S. Furthermore, let $PS\,(V)$ and $\overline{V}_{PS}$ be the value of the investment opportunity and the investment trigger for the private firm when under the subsidization policy, respectively. Noting that the only difference lies in the diminishing of the direct cost of investment, it is clear that they will be equal to:

$$PS(V) = \begin{cases} \left(\overline{V}_{PS}\,(1 - t_c) - \overline{X}_P + S\right)\left(\frac{V}{\overline{V}_{PS}}\right)^{\beta_1} & \text{for } V < \overline{V}_{PS} \\ V\,(1 - t_c) - \overline{X}_P + S \quad \text{for } V \ \geq \overline{V}_{PS} \end{cases} \tag{4.13}$$

and

$$\overline{V}_{PS} = \frac{\beta_1}{\beta_1 - 1}\frac{\overline{X}_P - S}{1 - t_c}. \tag{4.14}$$

Taking the expression for the investment trigger, it is then obvious that this will be lower than if the subsidy were not in place ($\overline{V}_{PS} < \overline{V}_P$ for any $S > 0$). It also makes sense within the problem under study, because the cost of investment is lower and, hence, the investment decision should be easier to make, which is translated into a lower investment trigger value.

It is also clear that the government's payoff (now Π_{GS}) will be reduced by the subsidy value:

$$\Pi_{GS} = \Pi_G - S = t_c V + t_n \lambda_P \overline{X}_P - S. \tag{4.15}$$

Nevertheless, this alternative is only useful for the government if (i) it is able to stimulate immediate investment; and (ii) it gives the government greater payoff than if the project were to be undertaken by the public sector. Taking the first point made, we are then thinking of an optimal subsidy, capable of reaping immediate investment by the private firm. Such a subsidy is one that ensures that $\overline{V}_{PS} = V$, and is therefore equal to:

$$S_{opt} = \overline{X}_P - \frac{\beta_1 - 1}{\beta_1} V(1 - t_c). \tag{4.16}$$

Now looking into the government's payoff, the fact that it must be greater than if the public sector were to be the one undertaking the project, means that there is a maximum subsidy that the government is willing to concede, which is the one that guarantees that:

$$\Pi_{GS} \geq V_g - \overline{X}_G + t_n \lambda_G \overline{X}_G, \tag{4.17}$$

and, therefore, through the simplification of expressions, the maximum subsidy will be equal to:

$$S_{max} = t_c V + t_n \left(\lambda_P \overline{X}_P - \lambda_G \overline{X}_G\right) - (V_g - \overline{X}_G). \tag{4.18}$$

Given these two points, it is then true that the decision to subsidize will follow the condition:

$$S_{max} \geq S_{opt}. \tag{4.19}$$

If it is verified, the government will grant a subsidy to invest. Otherwise, the subsidy would not prompt immediate investment, as its value would be above the maximum the government is willing to pay, given that it would actually be better off by doing the investment itself in that case.

4.2.4. Tax policy and combination of both tax and subsidy stimulus

An alternative to subsidizing the project is a reduction of the capital income tax rate (t_c). Similarly to the optimal subsidy, the optimal tax rate will be the one that ensures that $\overline{V}_P = V$. Hence, taking the expression for the investment trigger when the private firm undertakes the project, through simplification of expressions, one arrives at:

$$\bar{t}_{c_{opt}} = 1 - \frac{\beta_1}{\beta_1 - 1} \frac{\overline{X}_P}{V}. \tag{4.20}$$

Then the reduction that the government must put in place is equal to $(t_c - \bar{t}_{opt})$, in order to ensure that the trigger for private investment is achieved instantly and the private firm invests immediately. However, and again similarly to the subsidy policy, the government will have a minimum tax rate acceptable, since it will only go ahead with the tax rate reduction if this is a better alternative to immediate investment by the public sector instead. Hence, the minimum tax rate is one that ensures that:

$$\Pi_G \geq V_g - \overline{X}_G + t_n \lambda_G \overline{X}_G. \tag{4.21}$$

Again by simplifying the expression, we arrive at:

$$t_{c_{min}} = \frac{t_n \left(\lambda_G \overline{X}_G - \ \lambda_P \overline{X}_P\right) + V_g - \overline{X}_G}{V}. \tag{4.22}$$

REMARK 4.1. *Equation (4.22) corrects the typo in Barbosa et al. (2016, Equation 25).*

Hence, the decision to use a tax rate reduction strategy to stimulate immediate private investment is reliant on the following condition being true:

$$t_{c_{min}} \ \leq \bar{t}_{c_{opt}}. \tag{4.23}$$

Comparing the subsidy and tax policies, given that both the optimal subsidy and tax rate would lead to private investment, it is possible to conclude which one should be better. If the government chooses to use the optimal subsidy:

$$\Pi_{GS}(S = S_{opt}) = t_c V + t_n \lambda_P \overline{X}_P - \left(\overline{X}_P - \frac{\beta_1 - 1}{\beta_1} V(1 - t_c)\right). \tag{4.24}$$

If instead it uses the optimal tax rate,

$$\Pi_{GS}(t_c = \bar{t}_{c_{opt}}; S = 0) = \left(1 - \frac{\beta_1}{\beta_1 - 1}\frac{\overline{X}_P}{V}\right) V + t_n \lambda_P \overline{X}_P. \tag{4.25}$$

As demonstrated in Appendix D:

$$\Pi_{GS}\left(S = S_{opt}\right) > \Pi_{GS}(t_c = \bar{t}_{c_{opt}}; S = 0). \tag{4.26}$$

This means that it would always be better to choose a subsidy policy to promote investment.

Nevertheless, and although it is sub-optimal as has been shown, the government may opt to do a combination of these two alternatives, again because of fiscal constraints, since tax reductions do not demand an immediate payment. The model as it stands allows to determine the optimal tax rate for a given subsidy or vice-versa, in order to account for this possibility.

4.2.5. The threat of public competition

Another option to stimulate investment is for the government to reveal its intention to invest at the trigger value $\overline{V}_G$, creating pressure for the private firm, if the project reaps a positive present value. What this achieves is a reduction of the trigger to invest for the private firm, which is computed under the monopoly assumption. It is important to state that this policy is obviously only viable if the government's trigger to invest is, in fact, lower than the private firm's critical value, i.e.:

$$\frac{\overline{V}_G}{1 - \gamma_V} < \overline{V}_P. \tag{4.27}$$

Notice that the government's trigger of investment is measured with the private efficiency. This means that we are taking into consideration that the private firm looks at this investment trigger under its own efficiency, which prevents it from losing some of the pre-tax profit flow value. Hence, the investment trigger as viewed by the private firm, which is aware of this inefficiency, would be higher. Overall, the rational is that the private firm will look at the investment trigger and account for the inefficiencies existent in the public sector and be aware that, for itself and its level of efficiency, that trigger is possibly higher. The expression takes this into account and measures already the government's trigger to invest under the private firm's efficiency, making sure that the value is below the current private sector's investment critical value.

The reduction in the private firm's trigger to invest which is a result of this threat of competition is equivalent to a subsidy s that ensures that its new trigger to invest is equal to $\frac{\overline{V}_G}{1-\gamma_V}$, i.e.:

$$\begin{aligned} \overline{V}_{PS} &= \frac{\overline{V}_G}{1-\gamma_V} \\ \Leftrightarrow \frac{\beta_1}{\beta_1 - 1}\frac{\overline{X}_P - s}{1-t_c} &= \frac{\overline{V}_G}{1-\gamma_V}. \end{aligned} \tag{4.28}$$

Since $\overline{V}_G$ is given by equation (4.12), solving this expression in order to s yields:

$$s = \overline{X}_P - \frac{(1-t_c)\,(1-t_n\lambda_G)\,\overline{X}_G}{1-\gamma_V}. \tag{4.29}$$

Hence, this results in a reduction of the optimal subsidy, making it equal to a new value of:

$$S_{new} = S_{opt} - s = \frac{(1-t_c)\,(1-t_n\lambda_G)\,\overline{X}_G}{1-\gamma_V} - \frac{\beta_1 - 1}{\beta_1}V(1-t_c). \tag{4.30}$$

Remark 4.2. *Equations (4.29) and (4.30) correct the typos in Barbosa et al. (2016, Equations 31 and 32), respectively.*

4.3. Prevention of disinvestment - GBM

What we are considering in this section is the possibility that uncertainty leads to a drop on the project value such that the companies are compelled to terminate their investment (V dropping to below the trigger of disinvestment). What we argue is that, if the project is one which is important to the economy of the country, the government may prefer to either pick up the project itself or put in place policies that make it desirable for the firm to keep its investment.

Being that now we are looking upon a disinvestment opportunity (i.e., a put option), we must instead consider the following boundary conditions:

Boundary Condition I: $H(\infty) = 0$;
Boundary Condition II: Value-Matching Condition;
Boundary Condition III: Smooth Pasting Condition.

Boundary condition I is the opposite of what it was in section 4.2. If before it was worthless to have an option to invest in a project with value zero, see that an option to sell that same project is now very valuable. On the other hand, no company would be interested in selling a highly valuable project, hence why, if that is the case, the option to exit the operating state becomes worthless. The value-matching condition here states that by disinvesting in the project, the firm must receive a payoff which equals receiving the divestment proceeds and losing the value of the project. The smooth-pasting condition, like before, guarantees that $H(V)$ is continuous and smooth at the trigger value for disinvestment, making it the optimal point to do so. Like before, both the value-matching condition and the smooth-pasting condition are to be set in accordance with the specific case at hand.

To accommodate the new Boundary Condition I, it is necessary that $A_H = 0$, which means that the solution now becomes:

$$H(V) = B_H V^{\beta_2}. \tag{4.31}$$

B_H is a constant to be determined from the other two boundary conditions (along with $\underline{V}_H$). β_2 is the negative root of the fundamental quadratic equation, being equal to:

$$\beta_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0. \tag{4.32}$$

The derivation of equation (4.32) can also be found in Appendix A.

4.3.1. The private firm's decision to disinvest

The problem at hand is now slightly different to the one we have seen previously. The reasoning before was that, if the private firm did not choose to invest because the value of investment was below the trigger to do so, then the government would either invest itself or change the conditions of investment to stimulate the private firm to engage in the project. Now we assume that the private firm is already in the operating state and so, first, we must study if its trigger to disinvestment is such that it causes the firm to exit the project. If it is so, that is when the government acts: it either invests itself on the project or puts in place conditions that induce the private company to stay in the operating state.

This being so, we will first study the position of the private firm, meaning whether it is going to stay in the operating state or if the uncertainty conditions have led the value of the project to fall below its trigger to exit. The boundary conditions are then as follows:

Value-Matching Condition: $P(\underline{V}_P) = \underline{X} - \underline{V}_P(1 - t_c)$
Smooth-Pasting Condition: $P'(\underline{V}_P) = -(1 - t_c)$,

with $\underline{X}$ denoting the proceeds from the sale/dissolution of the project for the private company.

Given these, then the solution for the value of this disinvestment opportunity and its trigger, as derived in Appendix E, are as follows:

$$P(V) = \begin{cases} \left(\underline{X} - \underline{V}_P\left(1 - t_c\right)\right)\left(\frac{V}{\underline{V}_P}\right)^{\beta_2} & \text{for } V > \underline{V}_P \\ \underline{X} - V\left(1 - t_c\right) & \text{for } V \leq \underline{V}_P \end{cases} \tag{4.33}$$

and

$$\underline{V}_P = \frac{\beta_2}{\beta_2 - 1}\frac{\underline{X}}{1 - t_c}. \tag{4.34}$$

If the firm continues on with the project, then the government payoff is as it was on section 4.2.1.:

$$\Pi_G = t_c V + t_n \lambda_P \underline{X}_P. \tag{4.35}$$

If, instead, the value of the project falls below $\underline{V}_P$, then the firm exits the operating state and the government loses the profit from corporate taxes levied on the company pertaining to this project and it also loses the positive externalities of the investment on the economy and its impact on overall collection of taxes. Hence, the government loses all payoff from this project (gets a payoff of zero), if no other measure is taken.

As stated, one of the alternatives for the government if this is the case, is to invest itself on the project. That being the case, we are referring to section 4.2.2. of this dissertation, where we can find expressions (4.11) and (4.12) for the government's payoff and trigger value, respectively. One important caveat to this alternative, however, is determining if it is even optimal for the government to choose this option, meaning if $V_g \geq \overline{V}_G$ is true. If it is not, then the government must rely on another alternative. If it is true, then we must compute the new government payoff and compare to the payoff the government receives if the other alternative is instead put in place.

This alternative is the one which relies on tax reduction, to prevent the private company from disinvesting at all. In this case, seeing that in this thesis we are considering subsidies to investment, this subsidy is not a viable policy to use here. The investment has already been made by the company and so a subsidy to investment is unlikely to motivate firms to keep their investments (the same reasoning can be made for the threat of public competition).

However, a small remark must be made. We are only considering that either the firm is on the idle state and we wish to make it switch to the operating state (stimulation of investment) or it is on the operating state and we wish to make it stay as such (prevention of disinvestment). There is a case to be made that the firm may exit the operating state (as $V \leq \underline{V}_P$ occurs) and then, through policies of stimulation of investment, which include investment subsidies, be induced to go back to the operating state. This is not what we wish to accomplish when we talk about prevention of disinvestment. In other words, we are not considering the dynamic entry and exit strategies in the spirit of Dixit (1989) and Dias and Shackleton (2011). This is the reason why investment subsidies are not to be considered in this section. Seeing that we are only considering investment subsidies and reduction of taxes, then the government now must rely on the latter to prevent disinvestment.

4.3.2. Tax policy

In this case, the optimal subsidy, is one that ensures that, at least, $V = \underline{V}_P$, but ideally would be the one that guarantees that $V > \underline{V}_P$. The optimal is the former because that would be the one that results in the desired result and has the least expenditure for the government.

If this is the case, then defining $\underline{t}_{c_{opt}}$ as the optimal tax rate for the case under study:

$$\underline{t}_{c_{opt}} = 1 - \frac{\beta_2}{\beta_2 - 1} \frac{\underline{X}}{\overline{V}}. \tag{4.36}$$

The reasoning for a minimum tax rate still applies here and, in fact, its value will be the same. The government will only go ahead with the tax rate reduction if this is a better alternative to investing itself instead. This way, the minimum tax rate is still given by expression (4.22).

CHAPTER 5

Methodology - CEV model

5.1. Base model - CEV

In order to expand upon the model that has been explored before, we will then, as stated, work with a CEV framework. In order to derive the expressions with which we will work with now, we will rely on Dias and Nunes (2011), as well as Cox (1975) and Davydov and Linetsky (2001), among others. We maintain the same notation as in the previous chapter, being that the expression that defines each variable must be taken within the context of the chapter and subsection it is inserted in.

The important distinction that is introduced here comes from the diffusion process that governs the pre-tax profit flows of the project, which now will be:

$$dV = \alpha V dt + \sigma(V) V dz. \tag{5.1}$$

As before, $\alpha = r - q$ is the expected profit flows drift and dz is the increment of the Wiener process, being that we consider that we are already under the $\mathbb{Q}$ martingale probability measure. Therefore, z can also be defined as the standard Brownian Motion under $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \mathcal{F}_t : t \geq t_0$. The difference lies within the instantaneous volatility per unit of asset returns, which now is represented as the local volatility function $\sigma(V)$, which will be defined as:

$$\sigma(V) = \delta V^{\frac{\beta}{2}-1}, \tag{5.2}$$

for $\beta \in \mathbb{R}$ and $\delta \in \mathbb{R}^+$.

Taking this, we can write the diffusion process as the following stochastic differential equation:

$$dV = \alpha V dt + \delta V^{\frac{\beta}{2}} dz. \tag{5.3}$$

The model parameter δ is essentially a scale parameter that fixes the initial instantaneous volatility at time 0, i.e. by defining $\sigma(V_0)$ we can compute δ for different β levels. The β is the parameter that accounts for the leverage effect, being that $\beta < 2$ corresponds to a direct leverage effect and $\beta > 2$ to an indirect leverage effect. Notice that the GBM can be accommodated within this expression, when $\beta = 2$. This way, the methodology used within Barbosa et al. (2016) is, as stated in the Literature review of this dissertation, a specific case of the more general CEV framework.

Our work will be focused on the case of $\beta < 2$, which is consistent with the problem under study. Being that we are considering a project as the asset over which the analysis is performed, it is acceptable to expect a negative volatility skew, which, as mentioned in the Literature review and as is known, is most usual for stock index options (equity) and crude oil prices. Furthermore, the case of $\beta > 2$, would imply other complexities to our work. If this is the case, then the local volatility function generates an upward-sloping volatility skew and the discounted price process under the CEV is not a martingale, which can create *stock bubbles*[7].

Our approach to the methodology under a CEV model will be similar to the one we took in the previous chapter. Since we have already defined the diffusion process under CEV, we now solve the following ODE, which must incorporate the particularities that have been shown and discussed:

$$\frac{1}{2}\sigma^2 V^{\beta}\frac{d^2H(V)}{dV^2} + (r-q)\,V\frac{dH(V)}{dV} - rH(V) = 0. \tag{5.4}$$

The solution of this ODE (as per Davydov and Linetsky (2001), for example) depends on the specific case under consideration. That is, depends on whether β is smaller or greater than 2 and also if a drift exists or not. Seeing that we have already defined that we will only be working with $\beta < 2$, we only have to take into account two cases: i) $r = q$; or ii) $r \neq q$.

i. $r = q$

Considering the following auxiliary functions:

$$s \;:=\; \frac{1}{|\beta - 2\,|} \tag{5.5}$$

$$z\,(V) := \frac{2\sqrt{2r}}{\delta\,|\beta-2|}V^{1-\frac{\beta}{2}}, \tag{5.6}$$

then the solution of the ODE (5.4) is as follows:

$$H(V) = C_H V^{\frac{1}{2}} I_s\,(z(V)) + D_H V^{\frac{1}{2}} K_s(z(V)), \tag{5.7}$$

with C_H and D_H being two constants, and $I\,(\cdot)$ and $K(\cdot)$ being modified Bessel functions. This expression can be considered the equivalent of expression (4.3) in the previous chapter.

ii. $r \neq q$

Considering the following auxiliary functions:

$$x\,(V) := \frac{2\,|r-q|}{\delta^2\,|\beta-2|}V^{2-\beta} \tag{5.8}$$

$$\varepsilon := sign[(r-q)\,(\beta-2)] \tag{5.9}$$

[7] The explanation for this effect and its consequences, specifically under the CEV model, can be found in Dias et al. (2020).

$$k := \varepsilon\left(\frac{1}{2} + \frac{1}{2\left(\beta - 2\right)}\right) - \frac{r}{\left[\left(r - q\right)\left(\beta - 2\right)\right]} \tag{5.10}$$

$$m := \frac{1}{2\left|\beta - 2\right|} \tag{5.11}$$

$$a := \frac{1}{2} + m - k \tag{5.12}$$

$$b := 1 + 2m \tag{5.13}$$

$$y := \left(\frac{2\left|r - q\right|}{\delta^2\left|\beta - 2\right|}\right)^{\frac{b}{2}} \tag{5.14}$$

then the solution of ODE (5.4) is:

$$H(V) = C_H V^{\frac{\beta-1}{2}} e^{\frac{\varepsilon}{2}x(V)} M_{k,m}\left(x(V)\right) + D_H V^{\frac{\beta-1}{2}} e^{\frac{\varepsilon}{2}x(V)} W_{k,m}(x(V)), \tag{5.15}$$

where C_H and D_H are still constants, and $M(\cdot)$ and $W(\cdot)$ are Whittaker's functions.

5.2. Stimulation of investment - CEV

Once again we will consider the two alternatives explored in chapter 4, meaning that we will study both how the government can stimulate investment on relevant projects for the economy when necessary, and also how it can prevent disinvestment in those projects. Like before as well, we will begin by looking into the stimulation of investment.

Considering that investment is essentially a call option on a given project, just as in chapter 4 we will take the solution of the ODE (5.4), which in this case is dependent on the relation between r and q (expressions (5.7) and (5.15)) and apply the necessary boundary conditions. So, following the same reasoning as in chapter 4, it must be true that:

$$\lim_{V \to 0^+} H\left(V\right) = 0. \tag{5.16}$$

This being so, then it follows that, when V is below the trigger value and $r = q$,

$$H(V) = C_H V^{\frac{1}{2}} I_s\left(z(V)\right), \tag{5.17}$$

and when $r \neq q$

$$H(V) = C_H V^{\frac{\beta-1}{2}} e^{\frac{\varepsilon}{2}x(V)} M_{k,m}\left(x(V)\right). \tag{5.18}$$

For the latter, we will use the equivalent expression using Kummer's function (Abramowitz and Stegun, 1972, equation 13.1.32):

$$H(V) = C_H e^{\frac{(\varepsilon-1)x(V)}{2}} yVM\left(a,b,x(V)\right). \tag{5.19}$$

Now from each of the particular cases being considered, which will be the same as in the previous chapter, we will define the appropriate value-matching and smooth-pasting conditions and derive the resulting expressions.

5.2.1. The private firm undertakes the project

The boundary conditions for this case will be the same as before, and this will be true for all cases to be considered. As stated, the cases considered will be the same as in Section 4 as well as the parameters included in the model. This way, when considering that the private firm undertakes the project ($H = P$), we will still consider:

Value-Matching Condition: $P\left(\overline{V}_P\right) = \overline{V}_P\left(1-t_c\right) - \overline{X}_P$
Smooth-Pasting Condition: $P'\left(\overline{V}_P\right) = 1 - t_c$,

with $\overline{V}_P$ being the investment threshold for the private firm, like before.

i. $r = q$

If this is the case, then we can derive $P'(V)$ (Appendix F):

$$P'(V) = C_P V^{-\frac{1}{2}} I_s\left(z(V)\right) + C_P V^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(V)\right). \tag{5.20}$$

Then when taking into account the boundary conditions, we get the following system of equations:

$$\begin{cases} C_P \overline{V}_P^{\frac{1}{2}} I_s\left(z(\overline{V}_P)\right) = \overline{V}_P\left(1-t_c\right) - \overline{X}_P \\ C_P \overline{V}_P^{-\frac{1}{2}} I_s\left(z(\overline{V}_P)\right) + C_P \overline{V}_P^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(\overline{V}_P)\right) = 1 - t_c. \end{cases} \tag{5.21}$$

We can simplify further and obtain an expression for C_P and one from which we can numerically find $\overline{V}_P$ (Appendix G):

$$C_P = \frac{1}{I_s\left(z(\overline{V}_P)\right)} \overline{V}_P^{-\frac{1}{2}} (\overline{V}_P\left(1-t_c\right) - \overline{X}_P) \tag{5.22}$$

$$-\overline{X}_P + \overline{V}_P^{\frac{2-\beta}{2}} \left(\overline{V}_P\left(1-t_c\right) - \overline{X}_P\right) \frac{\sqrt{2r}}{\delta} \frac{I_{s+1}\left(z(\overline{V}_P)\right)}{I_s\left(z(\overline{V}_P)\right)} = 0. \tag{5.23}$$

ii. $r \neq q$

Now we derive $P'(V)$, as is shown in Appendix H:

$$\begin{aligned} &P'(V) = \\ &P(V) \times \left[V^{-1} + x'(V)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(V)\right)}{M\left(a,b,x(V)\right)}\right]\right]. \end{aligned} \tag{5.24}$$

Just as before, taking this expression into consideration, then the system of equations becomes:

$$\begin{cases} C_P e^{\frac{(\varepsilon-1)x(\overline{V}_P)}{2}} y\overline{V}_P M\left(a,b,x(\overline{V}_P)\right) = \overline{V}_P\ (1-t_c) - \overline{X}_P \\ P(\overline{V}_P) \times \left[\overline{V}_P^{\ -1} + x'(\overline{V}_P)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_P)\right)}{M\left(a,b,x(\overline{V}_P)\right)}\right]\right] = 1-t_c. \end{cases} \tag{5.25}$$

Again, via simplification of the system of equations, we are able to arrive to an expression for C_P and one that allows us to numerically find $\overline{V}_P$ (Appendix I):

$$C_P = \left(\overline{V}_P\ (1-t_c) - \overline{X}_P\right) \times \overline{V}_P^{\ -1} \times \frac{1}{e^{\frac{(\varepsilon-1)x(\overline{V}_P)}{2}} yM\left(a,b,x(\overline{V}_P)\right)} \tag{5.26}$$

and

$$\begin{aligned} &-\overline{X}_P + (\overline{V}_P\ (1-t_c) - \overline{X}_P)(2-\beta) \\ &\times x(\overline{V}_P)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_P)\right)}{M\left(a,b,x(\overline{V}_P)\right)}\right] = 0. \end{aligned} \tag{5.27}$$

Summarizing and referring back to expressions (5.17) and (5.19), after a few simplifications in the expressions, the private sector payoff will be:

$$P(V) = \begin{cases} \left(\overline{V}_P\,(1-t_c) - \overline{X}_P\right)\left(\frac{V}{\overline{V}_P}\right)^{\frac{1}{2}} \frac{I_s(z(V))}{I_s\left(z(\overline{V}_P)\right)} \text{ for } V < \overline{V}_P \ \ \wedge \ r = q \\ V\,(1-t_c) - \overline{X}_P \ \text{ for } V \geq \ \overline{V}_P \ \ \wedge \ \ r = q \\ \left(\overline{V}_P\ \ (1-t_c) - \overline{X}_P\right)\frac{V}{\overline{V}_P}\frac{e^{\frac{(\varepsilon-1)x(V)}{2}} yM(a,b,x(V))}{e^{\frac{(\varepsilon-1)x(\overline{V}_P)}{2}} yM\left(a,b,x(\overline{V}_P)\right)} \\ \qquad \text{for } V < \overline{V}_P \ \ \wedge \ r \neq q \\ V\,(1-t_c) - \overline{X}_P \ \text{ for } V \geq \overline{V}_P \ \ \wedge \ r \neq q. \end{cases} \tag{5.28}$$

Either in case i) or ii), and as before, the public sector payoff will be:

$$\Pi_G = t_c V + t_n \lambda_P \overline{X}_P. \tag{5.29}$$

Its value will also be the same as when considering a GBM diffusion process to model the evolution of the project value.

5.2.2. The public sector undertakes the project

If the public sector decides to be the one to undertake the project, the value-matching and smooth-pasting conditions will be:

Value-Matching Condition: $G\left(\overline{V}_G\right) = \overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G$
Smooth-Pasting Condition: $G'\left(\overline{V}_G\right) = 1,$

with $\overline{V}_G$ being the investment threshold for the private firm, like before.

We still consider public inefficiencies and, being so, we still have that:

$$\overline{X}_G = (1+\gamma_X)\,\overline{X}_P \tag{5.30}$$

and

$$V_g = (1-\gamma_V)\,V. \tag{5.31}$$

i. $r = q$

$G'(V)$ will actually be equivalent to $P'(V)$:

$$G'(V) = C_G V^{-\frac{1}{2}} I_s\left(z(V)\right) + C_G V^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(V)\right). \tag{5.32}$$

We get the following system of equations from the defined boundary conditions:

$$\begin{cases} C_G \overline{V}_G^{\frac{1}{2}} I_s\left(z(\overline{V}_G)\right) = \overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G \\ C_G \overline{V}_G^{-\frac{1}{2}} I_s\left(z(\overline{V}_G)\right) + C_G \overline{V}_G^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(\overline{V}_G)\right) = 1. \end{cases} \tag{5.33}$$

Again we simplify to find the expression for C_G under the new conditions and one from which we can numerically find $V\overline{V}_G$ (Appendix J):

$$C_G = \left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right) \overline{V}_G^{-\frac{1}{2}} \frac{1}{I_s\left(z(\overline{V}_G)\right)} \tag{5.34}$$

and

$$\left(-\overline{X}_G + t_n\lambda_G\overline{X}_G\right) + \overline{V}_G^{\frac{2-\beta}{2}} \left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right) \frac{\sqrt{2r}}{\delta} \frac{I_{s+1}\left(z(\overline{V}_G)\right)}{I_s\left(z(\overline{V}_G)\right)} = 0. \tag{5.35}$$

ii. $r \neq q$

Once again, $G'(V)$, will be equivalent to $P'(V)$:

$$G'(V) = G(V) \times \left[V^{-1} + x'(V)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(V)\right)}{M\left(a,b,x(V)\right)}\right]\right]. \tag{5.36}$$

Then the system of equations to consider becomes:

$$\begin{cases} C_G e^{\frac{(\varepsilon-1)x(\overline{V}_G)}{2}} y \overline{V}_G M\left(a,b,x(\overline{V}_G)\right) = \overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G \\ G(\overline{V}_G)\left[\overline{V}_G^{-1} + x'(\overline{V}_G)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_G)\right)}{M\left(a,b,x(\overline{V}_G)\right)}\right]\right] = 1. \end{cases} \tag{5.37}$$

The derivation of the expression for C_G and for the expression that we will use to numerically find $\overline{V}_G$ can be found in Appendix K:

$$C_G = \left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right) \times \overline{V}_G^{\,-1} \times \frac{1}{e^{\frac{(\varepsilon-1)x(\overline{V}_G)}{2}} yM\left(a, b, x(\overline{V}_G)\right)} \tag{5.38}$$

and

$$\begin{aligned} &- \overline{X}_G + t_n\lambda_G\overline{X}_G + \left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right)(2-\beta) \\ &\quad \times x(\overline{V}_G)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1, b+1, x(\overline{V}_G)\right)}{M\left(a, b, x(\overline{V}_G)\right)}\right] = 0. \end{aligned} \tag{5.39}$$

Summarizing and simplifying, whilst also considering V_g to account for public inefficiencies:

$$G(V) = \begin{cases} \left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right)\left(\frac{V_g}{\overline{V}_G}\right)^{\frac{1}{2}} \frac{I_s(z(V))}{I_s\left(z(\overline{V}_G)\right)} \text{ for } V_g < \overline{V}_G \quad \wedge \ r = q \\ V_g - \overline{X}_G + t_n\lambda_G\overline{X}_G \text{ for } V_g \geq \ \overline{V}_G \quad \wedge \ r = q \\ \left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right)\frac{V_g}{\overline{V}_G} \frac{e^{\frac{(\varepsilon-1)x(V_g)}{2}} yM(a,b,x(V_g))}{e^{\frac{(\varepsilon-1)x(\overline{V}_G)}{2}} yM\left(a,b,x(\overline{V}_G)\right)} \\ \qquad \text{for } V_g < \overline{V}_G \quad \wedge \ r \neq q \\ V_g - \overline{X}_G + t_n\lambda_G\overline{X}_G \text{ for } V_g \geq \ \overline{V}_G \quad \wedge \ r \neq q. \end{cases} \tag{5.40}$$

5.2.3. The government subsidizes the project and the optimal incentive policy

Once again, the consequence of implementing an investment subsidy policy is the diminishing of the costs of investment for the private sector. Like before, instead of having $\overline{X}_P$, the new cost of investing will be $\overline{X}_P - S$.

Hence, it is quite straightforward that:

$$PS(V) = \begin{cases} \left(\overline{V}_{PS}\left(1-t_c\right) - \overline{X}_P + S\right)\left(\frac{V}{\overline{V}_{PS}}\right)^{\frac{1}{2}} \frac{I_s(z(V))}{I_s\left(z(\overline{V}_{PS})\right)} \\ \qquad \text{for } V < \overline{V}_{PS} \quad \wedge \ r = q \\ V\left(1-t_c\right) - \overline{X}_P + S \ \text{ for } V \geq \ \overline{V}_{PS} \quad \wedge \quad r = q \\ \left(\overline{V}_{PS}\left(1-t_c\right) - \overline{X}_P + S\right)\frac{V}{\overline{V}_{PS}} \frac{e^{\frac{(\varepsilon-1)x(V)}{2}} yM(a,b,x(V))}{e^{\frac{(\varepsilon-1)x(\overline{V}_{PS})}{2}} yM\left(a,b,x(\overline{V}_{PS})\right)} \\ \qquad \text{for } V < \overline{V}_{PS} \quad \wedge \ r \neq q \\ V\left(1-t_c\right) - \overline{X}_P + S \ \text{ for } V \geq \ \overline{V}_{PS} \quad \wedge \ r \neq q. \end{cases} \tag{5.41}$$

To find the trigger value numerically, and using the same reasoning, we will use the following equations, being that for $r = q$ we have:

$$-\overline{X}_P + S + \overline{V}_{PS}^{\ \frac{2-\beta}{2}}\left(\overline{V}_{PS}\left(1-t_c\right) - \overline{X}_P + S\right)\frac{\sqrt{2r}}{\delta}\frac{I_{s+1}\left(z(\overline{V}_{PS})\right)}{I_s\left(z(\overline{V}_{PS}\right)} = 0 \tag{5.42}$$

and for $r \neq q$:

$$
\begin{aligned}
&-\overline{X}_P + S + (\overline{V}_{PS}\ (1-t_c) - \overline{X}_P + S)(2-\beta) \\
&\times x(\overline{V}_{PS}) \left[\frac{(\varepsilon-1)}{2} + \frac{a}{b} \frac{M\left(a+1, b+1, x(\overline{V}_{PS})\right)}{M\left(a, b, x(\overline{V}_{PS})\right)} \right] = 0.
\end{aligned} \tag{5.43}
$$

When it comes to the government payoff, like before, it will be reduced by the amount of the subsidy:

$$
\Pi_{GS} = \Pi_G - S = t_c V + t_n \lambda_P \overline{X}_P - S. \tag{5.44}
$$

From expressions (5.42) and (5.43), we are able to derive the expressions for the optimal subsidy under $r = q$ and $r \neq q$. That is, we can find S_{opt} by setting $\overline{V}_{PS} = V$ in expression (5.42) and (5.43) respectfully. Then, we just solve the expressions in order to S, meaning that, for $r = q$:

$$
S_{opt} = \frac{\overline{X}_P - V^{\frac{2-\beta}{2}} \left(V\,(1-t_c) - \overline{X}_P\right) \frac{\sqrt{2r}}{\delta} \frac{I_{s+1}(z(V))}{I_s(z(V))}}{1 + V^{\frac{2-\beta}{2}} \frac{\sqrt{2r}}{\delta} \frac{I_{s+1}(z(V))}{I_s(z(V))}} \tag{5.45}
$$

and for $r \neq q$:

$$
S_{opt} = \frac{\overline{X}_P - (V\,(1-t_c) - \overline{X}_P)(2-\beta)x(V) \left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M(a+1,b+1,x(V))}{M(a,b,x(V))}\right]}{1 + (2-\beta)x(V) \left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M(a+1,b+1,x(V))}{M(a,b,x(V))}\right]}. \tag{5.46}
$$

The maximum subsidy is still as previously defined, for the same reasons:

$$
S_{max} = t_c V + t_n (\lambda_P \overline{X}_P - \lambda_G \overline{X}_G) - (V_g - \overline{X}_G). \tag{5.47}
$$

5.2.4. Tax policy and combination of both tax and subsidy stimulus

To find the optimum tax rate (and therefore optimum tax reduction) we once again run into the difficulty of not having a direct expression for the trigger value. Now we are assuming that there is no subsidy, and so, similarly to what we did for the optimum subsidy, we will use expressions (5.23) and (5.27), setting $V = \overline{V}_P$ and solving these expressions in order to t_c. This is how we will find $t_{c_{opt}}$ for $r = q$ and $r \neq q$. The optimal tax reduction will therefore be $(t_c - t_{c_{opt}})$. Hence, for $r = q$ we have:

$$
\bar{t}_{c_{opt}} = -\frac{\overline{X}_P - V^{\frac{2-\beta}{2}} \left(V - \overline{X}_P\right) \frac{\sqrt{2r}}{\delta} \frac{I_{s+1}(z(V))}{I_s(z(V))}}{V^{\frac{4-\beta}{2}} \frac{\sqrt{2r}}{\delta} \frac{I_{s+1}(z(V))}{I_s(z(V))}} \tag{5.48}
$$

and for $r \neq q$:

$$
\bar{t}_{c_{opt}} = -\frac{\overline{X}_P - (V\ \ - \overline{X}_P)(2-\beta)x(V) \left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M(a+1,b+1,x(V))}{M(a,b,x(V))}\right]}{V(2-\beta)x(V) \left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M(a+1,b+1,x(V))}{M(a,b,x(V))}\right]}. \tag{5.49}
$$

The minimum tax rate will be the same as before:

$$t_{c_{min}} = \frac{t_n \left(\lambda_G \overline{X}_G - \ \lambda_P \overline{X}_P\right) + V_g - \overline{X}_G}{V}. \tag{5.50}$$

The comparison of tax and subsidy policy is not as straightforward as it was under a GBM, as is also the case for considering a scenario which considers both. This being the case, we will not be able to conclude on which is the better option for the government to use, as we have in chapter 4.2.4..

5.2.5. The threat of public competition

We follow the same reasoning as previously, remembering that the effect under consideration here comes from creating pressure on the private sector, which causes a similar reaction to when a subsidy is implemented. The value of this subsidy, as shown before, is such that it guarantees that, for $r = q$ or $r \neq q$:

$$\overline{V}_{PS} = \frac{\overline{V}_G}{1 - \gamma_V}. \tag{5.51}$$

We find the value of this equivalent subsidy via expressions similar to (5.45) and (5.46) when defining the trigger value as in the expression above. This way we can find a subsidy s which decreases the value of the optimum subsidy. For either $r = q$ or $r \neq q$, we will have:

$$S_{new} = S_{opt} - s. \tag{5.52}$$

5.3. Prevention of disinvestment - CEV

Moving on to considering the prevention of disinvestment in important projects for the economy, once again we follow the same reasoning as when dealing with the GBM. Disinvestment is, as explained before, comparable to a put option on a project. It must be true that:

$$\lim_{V \to \infty} H\left(V\right) = 0. \tag{5.53}$$

This being so, then it follows that, when V is above the trigger value and $r = q$:

$$H\left(V\right) = D_H V^{\frac{1}{2}} K_s(z(V)) \tag{5.54}$$

and when $r \neq q$:

$$H\left(V\right) = D_H V^{\frac{\beta-1}{2}} e^{\frac{\varepsilon}{2} x(V)} W_{k,m}(x(V)). \tag{5.55}$$

Again, for ease, for the latter expression we will opt for the equivalent expression using Kummer's function (Abramowitz and Stegun, 1972, equation 13.1.33):

$$H\left(V\right) = D_H e^{\frac{(\varepsilon-1)x(V)}{2}} yVU\left(a, b, x(V)\right). \tag{5.56}$$

5.3.1. The private firm's decision to disinvest

This section will be equivalent to section 4.3.1. This way, we will now focus on the methodology and not on the reasoning, since it has been explored before. Therefore, we will set the appropriate boundary conditions:

Value-Matching Condition: $P\left(\underline{V}_P\right) = \underline{X} - \underline{V}_P\left(1 - t_c\right)$
Smooth-Pasting Condition: $P'\left(\underline{V}_P\right) = -\left(1 - t_c\right)$,

with $\underline{V}_P$ being again the disinvestment threshold for the private firm.

i. $r = q$

If this is the case, then we can derive $P'\left(V\right)$, following a reasoning similar to the one on Appendix F:

$$P'\left(V\right) = \ D_P V^{-\frac{1}{2}} K_s\left(z(V)\right) - D_P V^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} K_{s+1}\left(z(V)\right). \tag{5.57}$$

Then, when taking into account the boundary conditions, we get the following system of equations:

$$\begin{cases} D_P \underline{V}_P^{\frac{1}{2}} K_s\left(z(\underline{V}_P)\right) = \underline{X} - \underline{V}_P\ \left(1 - t_c\right) \\ D_P \underline{V}_P^{-\frac{1}{2}} K_s\left(z(\underline{V}_P)\right) - D_P \underline{V}_P^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} K_{s+1}\left(z(\underline{V}_P)\right) = -\left(1 - t_c\right). \end{cases} \tag{5.58}$$

We can simplify further and obtain an expression for D and one from which we can numerically find $V^*_{P_{CEV\_disinv\, r=q}}$ (Appendix L):

$$D_P = \frac{1}{K_s\left(z(\underline{V}_P)\right)} \underline{V}_P^{-\frac{1}{2}} \left(\underline{X} - \underline{V}_P\ \left(1 - t_c\right)\right) \tag{5.59}$$

and

$$\underline{X} - \underline{V}_P^{\frac{2-\beta}{2}} \left(\underline{X} - \underline{V}_P\ \left(1 - t_c\right)\right) \frac{\sqrt{2r}}{\delta} \frac{K_{s+1}\left(z(\underline{V}_P)\right)}{K_s\left(z(\underline{V}_P)\right)} = 0. \tag{5.60}$$

ii. $r \neq q$

We derive $P'\left(V\right)$, now following a reasoning similar to the one on Appendix H, with the derivative of $U(a, b, x(V))$ taken from Abramowitz and Stegun (1972), equation 13.4.21.:

$$\begin{aligned} &P'\left(V\right) = \\ &P\left(V\right) \times \left[V^{-1} + x'(V)\left[\frac{(\varepsilon - 1)}{2} - a\frac{U\left(a+1, b+1, x(V)\right)}{U\left(a, b, x(V)\right)}\right]\right]. \end{aligned} \tag{5.61}$$

Just as before, taking this expression into consideration, then the system of equations becomes:

$$
\begin{cases}
D_P e^{\frac{(\varepsilon-1)x(\underline{V}_P)}{2}} y \underline{V}_P U\left(a,b,x(\underline{V}_P)\right) = \underline{X} - \underline{V}_P\left(1-t_c\right) \\
P\left(\underline{V}_P\right) \\
\quad \times \left[\underline{V}_P{}^{-1} + x'(\underline{V}_P)\left[\frac{(\varepsilon-1)}{2} - a\frac{U(a+1,b+1,x(\underline{V}_P))}{U(a,b,x(\underline{V}_P))}\right]\right] = -\left(1-t_c\right).
\end{cases}
\tag{5.62}
$$

From the derivations made in Appendix M we are able to arrive to an expression for D and one to numerically find $\underline{V}_P$:

$$
D_P = \left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right)\underline{V}_P{}^{-1}\frac{1}{e^{\frac{(\varepsilon-1)x(\underline{V}_P)}{2}} y U\left(a,b,x(\underline{V}_P)\right)}
\tag{5.63}
$$

and

$$
\begin{aligned}
&\underline{X} + \left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right)(2-\beta) \\
&\quad \times x\left(\underline{V}_P\right)\left[\frac{(\varepsilon-1)}{2} + -a\frac{U\left(a+1,b+1,x(\underline{V}_P)\right)}{U\left(a,b,x(\underline{V}_P)\right)}\right] = 0.
\end{aligned}
\tag{5.64}
$$

Summarizing, the private sector payoff will be:

$$
P_{CEV\_disinv}(V) =
\begin{cases}
\left(\underline{X} - \underline{V}_P\ \left(1-t_c\right)\right)\left(\frac{V}{\underline{V}_P}\right)^{\frac{1}{2}} \frac{K_s(z(V))}{K_s(z(\underline{V}_P))} \\
\qquad \text{for } V > \underline{V}_P \quad \wedge \ r = q \\
\underline{X} - V\left(1-t_c\right) \quad \text{for } V \leq \ \underline{V}_P \quad \wedge \quad r = q \\
\left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right)\frac{V}{\underline{V}_P}\frac{e^{\frac{(\varepsilon-1)x(V)}{2}}U(a,b,x(V))}{e^{\frac{(\varepsilon-1)x(\underline{V}_P)}{2}}U(a,b,x(\underline{V}_P))} \\
\qquad \text{for } V > \underline{V}_P \ \wedge \ r \neq q \\
\underline{X} - V\left(1-t_c\right) \quad \text{for } V \leq \underline{V}_P \ \wedge \ r \neq q
\end{cases}
\tag{5.65}
$$

Just as explained before, if the private company does decide to disinvest, then the government payoff drops to zero. The government can then either pick up the investment itself, with payoff as given by equation (5.40), or it can set up a tax reduction to prevent the private company from disinvesting.

5.3.2. Tax policy

To find the optimum tax rate (and therefore optimum tax reduction) we will now use expressions (5.60) and (5.64), setting $\underline{V}_P = V$ and solving in order to t_c. This is how we will find $\underline{t}_{c_{opt}}$ and so the optimal tax reduction will therefore be $(t_c - \underline{t}_{c_{opt}})$. The expression for when $r = q$ will therefore be:

$$
\underline{t}_{c_{opt}} = \frac{\underline{X} - \left(\underline{X} - V\ \right)V^{\frac{2-\beta}{2}}\frac{\sqrt{2r}}{\delta}\frac{K_{s+1}(z(V))}{K_s(z(V))}}{V^{\frac{4-\beta}{2}}\frac{\sqrt{2r}}{\delta}\frac{K_{s+1}(z(V))}{K_s(z(V))}}
\tag{5.66}
$$

and when $r \neq q$:

$$\underline{t}_{C_{opt}} = -\frac{\underline{X} + (\underline{X} - V)\,(2-\beta)x\,(V)\left[\frac{(\varepsilon-1)}{2} - a\frac{U(a+1,b+1,x(V))}{U(a,b,x(V))}\right]}{V(2-\beta)x\,(V)\left[\frac{(\varepsilon-1)}{2} - a\frac{U(a+1,b+1,x(V))}{U(a,b,x(V))}\right]}. \tag{5.67}$$

The minimum tax rate will be the same as before, given by expression (4.22).

CHAPTER 6

Application of the models

In order to better understand the dynamics of the problem at hand and study the implementation of the models we have derived, following what was done by Barbosa et al. (2016), we will use an example rooted in the Portuguese economy. This way, we will not only be able to draw conclusions about the models themselves but also compare our GBM results directly with the ones obtained by Barbosa et al (2016).

We will use MATLAB in order to obtain our results and the inputs to our model are summarized in Table 6.1. Regarding the public inefficiency parameters, we use the values provided by Afonso et al. (2005), for the investment multipliers we draw from Afonso and St. Aubyn (2009), and for the corporate tax rate and average tax rate we follow Eurostat (2021), which contains the most recent data published by Eurostat, relative to 2019. Specifically, when it comes to the average tax rate, we will compute it via summing the tax rate on labor (20.6%) to the tax rate on consumption net of labor taxes ($0.167 \times (1 - 0.206) = 13.3\%$). Regarding the values of current pre-tax gross project value and volatility we will use the ones assumed by Barbosa et al. (2016). The disinvestment proceeds will be taken as 50% of the project value, but a sensitivity analysis will be performed.

Regarding the dividend yield and risk-free rate, we will consider 3 cases, as specified in Table 6.2. Case 1 assumes the same values as Barbosa et al. (2016), where $r > q$. Cases 2 and 3 specify the two alternatives, $r = q$ and $r < q$ respectively.

TABLE 6.1. Inputs for the models.

Parameter	Value	Description
γ_X	0.266	Public inefficiency for undertaking the project
γ_V	0.300	Public inefficiency for running the project
λ_P	1.252	Multiplier of private investment
λ_G	0.835	Multiplier of public investment
t_c	0.315	Profit income tax rate
t_n	0.339	Average tax rate on the economy
$\overline{X}_P$	100	Investment cost for the private firm
V	300	Current pre-tax gross project value
σ	0.15	Instantaneous volatility of V
$\underline{X}$	150	Disinvestment Proceeds

Table 6.2. Cases for the risk-free interest rate and the dividend yield.

	Value of r	Value of q
Case 1	0.03	0.02
Case 2	0.03	0.03
Case 3	0.03	0.04

6.1. Results for the stimulation of investment

Table 6.3. Results for stimulation of investment, under GBM.

Output	Value of Barbosa et al. (2016)	Value if $r > q$ (Case 1)	Value if $r = q$ (Case 2)	Value if $r < q$ (Case 3)
$\overline{V}_P$	345.1	357.71	266.85	225.72
$\overline{V}_G$	215.1[8]	222.40	165.91	140.34
S_{max}	10.8	17.71	17.71	17.71
S_{opt}	13.1	16.13	-12.42	-32.91
$t_{c_{min}}$	0.2540	0.2560	0.2560	0.2560
$\bar{t}_{c_{opt}}$	0.183	0.1832	0.3907	0.4846
S_{new}	2.1	4.95	-23.60	-44.09

The results obtained taking GBM as the process that governs the behavior of the stochastic variable are presented in Table 6.3 and the ones for the CEV model can be found in Table 6.4. Taking the values of V and V_g that were assumed (V_g is implicitly equal to 210 given the parameters defined), we compare these to the values of thresholds that we have obtained and we also take the opportunity to compare our results with the ones in Barbosa et al. (2016), when observing case 1. One important aspect to explore further, however, is the relation between r and q, which, as we can see by the results obtained for the three different cases, is relevant. We interpret the dividend yield (q) as being the cost of waiting to exercise the option. This value is specific to the project itself. Notice that, between the three cases, what we have assumed is that the risk-free rate is the same and the cost of waiting is the one that increases from case 1 to case 3. Given this, it is to be expected that the threshold to trigger investment diminishes from case 1 to case 3, as the risk-free rate is the same but waiting becomes costlier.

Let us first look more closely to the results obtained under GBM. We can see that, comparing with the results obtained by Barbosa et al. (2016), we now have obtained a higher investment trigger value, meaning that exercising the option to invest is more difficult. The values that we have changed in relation to the parameters assumed by

[8]This value corrects the one presented by Barbosa et al. (2016).

Table 6.4. Results for stimulation of investment, under CEV.

	Value if $r > q$ (Case 1)					
Value of β	$\overline{V}_P$	$\overline{V}_G$	S_{opt}	$\bar{t}_{c_{opt}}$	S_{new}	$\frac{\overline{V}_G}{1-\gamma_V}$
1	370.43	266.01	25.89	0.0757	29.50	380.02
0	375.89	294.78	35.53	-0.0625	57.95	421.12
-1	376.35	311.90	44.21	-0.2279	87.48	445.58
-3	371.19	326.16	56.68	-0.5813	136.17	465.95
-6	361.61	330.86	67.84	-1.1300	175.66	472.66
	Value if $r = q$ (Case 2)					
Value of β	$\overline{V}_P$	$\overline{V}_G$	S_{opt}	$\bar{t}_{c_{opt}}$	S_{new}	$\frac{\overline{V}_G}{1-\gamma_V}$
1	297.19	213.05	-1.32	0.3239	2.05	304.35
0	318.59	252.34	11.09	0.2295	37.06	360.48
-1	330.15	276.76	22.09	0.1208	73.63	395.37
-3	338.50	300.71	38.47	-0.1132	140.39	429.59
-6	339.14	313.19	53.65	-0.4778	211.32	447.41
	Value if $r < q$ (Case 3)					
Value of β	$\overline{V}_P$	$\overline{V}_G$	S_{opt}	$\bar{t}_{c_{opt}}$	S_{new}	$\frac{\overline{V}_G}{1-\gamma_V}$
1	261.10	188.02	-21.30	0.4353	-17.26	268.60
0	286.98	227.78	-8.90	0.3710	17.78	325.40
-1	303.93	255.69	3.29	0.2917	57.44	365.27
-3	319.51	284.94	22.28	0.1186	131.95	407.05
-6	325.97	302.02	40.60	-0.1532	211.35	431.45

Barbosa et al. (2016) are only the average tax rate on the economy and the profit income tax rate. The value of the latter has increased, which then justifies why the investment is less desirable for the private firm and, therefore, leaving the idle state should not be as easy. This also explains why both the optimal subsidy and tax reduction are more costly for the government, since a larger incentive is needed. The subsidy that stimulates immediate investment if the government acts as a competitor has also increased, for the same reasons. For the government, since the average tax rate on the economy has a lower value, then the amount that the government gains via stimulation of the economy when investing itself is lower. Therefore, the option to invest must be more difficult to exercise.

When it comes to the maximum subsidy possible for the government, its value has increased in such a way that it is now possible to implement the subsidy incentive (which it was not under the values obtained by Barbosa et al. (2016), since the maximum subsidy was below the optimal). Seeing that more revenue is collected if the private firm goes through with the project, then the maximum that the government is able to provide in order to make that happen increases as well. As for the minimum tax rate, it has increased only slightly, and the option of using taxes as an incentive is still one that the government should not pursue, as in Barbosa et al. (2016), since the optimal tax rate is still below the minimum.

Now, comparing only along the three cases for the relationship between r and q considered, we see that the firm invests without any incentive in cases 2 and 3. Only if $r > q$ would the government need to provide incentives, and that is the only option since it is also not induced to exercise the option to invest itself. This is also why for cases 2 and 3 the value of the optimal subsidy or the subsidy if the government acts as a competitor for the project are negative, as they actually do not need to be implemented (the negative values can be interpreted as a tax that the private firm would have to pay to invest). If this is the case, then, as stated before, the only option available is to use an investment subsidy, for which the optimal value falls below the maximum (the optimal tax is below the minimum tax, hence it is not a viable policy to implement). The value for the government payoff, under GBM and for the two scenarios, is resumed in Table 6.5. We can see that for cases 2 and 3 the payoffs are the same for the government and that the better option is to let the private firm invest without any incentives in place (which, as stated, are not necessary). In case 1, the payoff of the government, will be equal to $\Pi_G - S_{opt}$, since it is possible to implement the optimal subsidy, and that is the only way to achieve investment in the project.

TABLE 6.5. Government payoff for different scenarios, under GBM.

Output if:	Value if $r > q$ (Case 1)	Value if $r = q$ (Case 2)	Value if $r < q$ (Case 3)
No incentives granted (Π_G)	Private firm does not invest without incentive	136.94	136.94
Possible investment from gov. ($G(V)$)	Government does not invest	Government invests with payoff 119.24	Government invests with payoff 119.24
Subsidy is granted(Π_{GS})	120.81	Not necessary	Not necessary
Tax cut is granted	Not viable	Not viable	Not viable

When it comes to the results for the CEV model, we remember that the values for the maximum subsidy and minimum tax are still as presented on Table 6.3. The first aspect that we can notice is that what we have stated regarding the progression from case 1 to case 3 still holds. That is, whatever the value of β we are considering, it is always true that moving from case 1 to case 3, as the risk-free rate is maintained constant and the cost of waiting increases, the value of the threshold of investment decreases. Secondly, we can see that in all instances $\frac{\overline{V}_G}{1-\gamma_V} > \overline{V}_P$, meaning that the private firm realizes that the public sector does not represent any real competition, which in turn means that competitive pressure from the government will never be a viable strategy (hence why the value of S_{new} is always greater than S_{opt}).

Simultaneously analyzing the results under CEV and comparing them with the ones obtained for the GBM, we can see that, overall, the trigger values under CEV are always greater and that they tend to increase as the value of β gets lower. Considering that the investment option that we are working with is deep in-the-money, this observation is in line with the results obtained by Dias and Nunes (2011), presented in their Figures 1, 2 and 3. Taking this, we see that assuming the GBM model leads to an underestimation of the effort that the government must do in order to stimulate immediate investment. This can also be shown by the fact that immediate investment is always possible under GBM (even though it requires incentives for case 1), whereas under CEV it is most of the time not optimally possible to achieve. More specifically, see that, similarly to the GBM results, under Case 1 the private firm does not invest without incentives, for any value of β considered. We see that for cases 2 and 3 it does, but also only for some β values (if $\beta = 1$ for case 2 and if $\beta = 1$ or 0 for case 3). Much like what we saw happening in our GBM results, in these cases we obtain a negative optimal subsidy, as indication that, as the private firm wants to invest without needing any incentive from the government, it would even be willing to pay the government for the opportunity to invest. A similar reasoning can be made when we look into the cases in which we have obtained a negative optimal tax rate. In such instances the opposite is true, meaning that in these scenarios the investment threshold is so high that the government would have to be willing to give money to the company for its profits, instead of taxing them. Unlike our GBM results, however, in case 1 the government is not able to stimulate immediate investment using any of the alternatives studied, for any value of β.

In Table 6.6. we present a synthesis of the government payoffs for the various scenarios, under the CEV model. In most cases the private firm does not invest without incentives, as discussed, and the government is not compelled to invest either, with both policies studied not being able to be put in place in an optimal manner. One particular scenario, however, is the one for $\beta = -1$ on case 3, where we see that neither the private firm nor the government invest immediately, but that the government can stimulate investment from the private sector using either one of the policies considered. See, from Table 6.6., that the payoff of the public sector is greater if it chooses to use the investment subsidy rather than the tax cut. This may show that the result from Barbosa et al. (2016) that it would always be better to choose a subsidy policy to promote investment still holds under CEV, which makes sense, seeing that an investment subsidy is a one-off expense to the government whereas a tax cut is a loss in revenue over time.

Table 6.6. Government payoff for different scenarious, under CEV.

Value of β	No incentives granted (Π_G)	Possible investment from gov. ($G(V)$)	Subsidy is granted(Π_{GS})	Tax cut is granted
Value if $r > q$ (Case 1)				
1, 0, -1, -3 or -6	Private firm does not invest without incentive	Government does not invest	Not viable	Not viable
Value if $r = q$ (Case 2)				
1	136.94	Government does not invest	Not necessary	Not necessary
0	Private firm does not invest without incentive	Government does not invest	125.85	Not viable
-1, -3 or -6	Private firm does not invest without incentive	Government does not invest	Not viable	Not viable
Value if $r < q$ (Case 3)				
1	136.94	119.24	Not necessary	Not necessary
0	136.94	Government does not invest	Not necessary	Not necessary
-1	Private firm does not invest without incentive	Government does not invest	133.65	125.95
-3 or -6	Private firm does not invest without incentive	Government does not invest	Not viable	Not viable

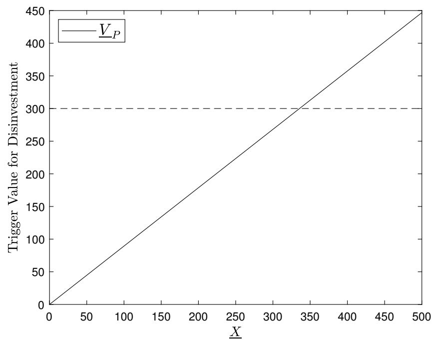
6.2. Results for the prevention of disinvestment

Table 6.7. Results for prevention of disinvestment, under GBM.

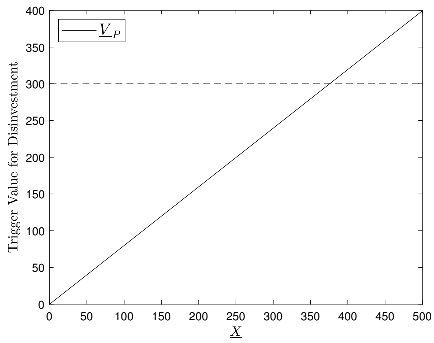
Output	Value if $r > q$ (Case 1)	Value if $r = q$ (Case 2)	Value if $r < q$ (Case 3)
$\underline{V}_P$	134.05	119.80	106.22
$\underline{t}_{c_{opt}}$	0.6939	0.7265	0.7575

For the study of prevention of disinvestment, we will look into the trigger value to exercise this option for the private firm and analyze how that decision is affected by the parameters and model that we are considering. Firstly, just as before, we see that the relationship between r and q has importance. Now q can be perceived as the dividend yield of the project and not as the cost of waiting, seeing that the company is already in the operating state and we are considering the option to leave the project. Hence, now the reasoning is opposite to what it was before. Because the value of the risk-free rate stays constant but the dividend yield increases from case 1 to case 3, then the value of the trigger to disinvest becomes lower from case 1 to case 3, making it more difficult to leave the operating state and give up the dividend yield that the company gains when operating. This effect can be seen in the results in Tables 6.7. and 6.8., under GBM and the CEV model, respectively.

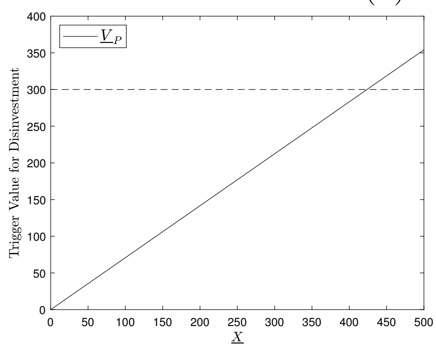
When specifically analysing the results under GBM, we see that the company is not motivated to disinvest given the conditions assumed and, more specifically, the given value of 150 for the proceeds of disinvestment. This is true for any of the three cases considered and it is the reason why the optimal tax rate for all three of them has such a high value. Overall, and since the firm is not compelled to disinvest, no policies are needed to prevent it from doing so. In fact, only if the proceedings from disinvestment were to rise above the current project value would that be a possibility, as we can see from the sensitivity analysis presented in Figure 6.1, for each of the cases considered.



(A) Case 1



(B) Case 2



(C) Case 3

FIGURE 6.1. Sensitivity analysis of the trigger value for disinvestment to $\underline{X}$, under GBM.

When it comes to the results under CEV, for simplification, we only include the values for $\beta = 1$ and $\beta = 0$. We see that as the value of β gets lower, so does the value of the disinvestment threshold. This being so, the conclusion remains as it was when considering the results under GBM and, hence, the government should not be worried that the firm considers disinvesting, at least for the assumed value of disinvestment proceeds. Again we see that the value of optimal tax is very high, as we have seen in the GBM results. Nevertheless, the threshold is what determines the firm's decision to disinvest and it is far below the current pre-tax gross project value. Hence, the government does not need to implement a tax cut to prevent it from happening.

Overall, it seems that wrongly considering a GBM instead of a CEV model to decide on the optimal tax policy to prevent disinvestment results once again in an unsuccessful outcome, as the effort to achieve it would have to be greater because the private firm's disinvestment threshold under CEV is lower. Hence, if it comes to the case that there is the risk of disinvesting, the tax cut would have to be larger if compared with the GBM, as the threshold at which that becomes a concern is lower.

TABLE 6.8. Results for prevention of disinvestment, under CEV.

	Value if $r > q$ **(Case 1)**		**Value if** $r = q$ **(Case 2)**		**Value if** $r < q$ **(Case 3)**	
Value of β	$\underline{V}_P$	$\underline{t}_{c_{opt}}$	$\underline{V}_P$	$\underline{t}_{c_{opt}}$	$\underline{V}_P$	$\underline{t}_{c_{opt}}$
1	108.13	0.6696	91.86	0.7060	77.95	0.7424
0	58.27	0.6516	35.27	0.6899	16.61	0.7315

CHAPTER 7

Conclusions

In this dissertation we have studied how the incentive policies put in place by the government can influence private firms' decisions to invest or disinvest. More specifically, we have considered subsidies to investment and tax cuts as two of these policies and calculated the optimal value that would either stimulate immediate investment or prevent disinvestment from the private sector. Our work is an extension of the work of Barbosa et al. (2016), who have studied only the issue of stimulation of investment and considered the GBM as the process which governs the pre-tax profit flows from any one project that the government considers to be important for the economy.

The contribution from our extensions to Barbosa et al. (2016) is two-fold. Firstly, as stated, our work considers both the possibility of the private firm being on the idle state and the government wishing that a relevant project be picked up to stimulate the economy, and also the possibility that the company is on the operating state regarding such a project, but considering disinvesting as the economic conditions deteriorate in periods of greater uncertainty. The latter scenario is one that was not considered in Barbosa et al. (2016), and one that is not treated often in Real Options literature. Hence, this is a gap that has been corrected by our work within this dissertation. Secondly, we go further with the work presented by the authors and expand it into a CEV model, for both scenarios mentioned. This being the case, we generalize the model via inclusion of more realistic characteristics (non-constant volatility), with GBM being a specific case (when the CEV-exponent is equal to 2).

Our contributions are relevant as they aim to explore the impact of the government making decisions regarding its incentive policy using a GBM model (perhaps for ease of tractability), when the true generation process is one aligned with the CEV model (with CEV-exponent being lower than 2). If that is the case, the policy put in place by the government may not reap the desired outcomes. In fact, taking into account our results, we see that that is the case both for the stimulation of immediate private investment and prevention of disinvestment. We have shown that in the former the threshold for investment under CEV is greater than if a GBM is considered, and in the latter the threshold for disinvestment is lower. Hence, in either case, considering a GBM instead of a CEV model would lead to an underestimation of the effort that has to be done in order to put in place policies that reap the desired outcome. Moreover, we have also seen from our application that when considering a CEV model, and especially for stimulation of investment, the use of incentive policies to do so is much more limited, with there being several cases in which stimulation of immediate investment is not possible for the government to achieve at all.

Nevertheless, our work can be improved upon. Much like we argue that there is an impact if a GBM process is wrongly assumed by the government in its decision-making process for incentive policies, it can be argued that there is also an impact if the CEV model is not the true generation process for the stochastic variable. This can be the case if there are mean-reverting properties at play or if the true CEV-exponent value is higher than 2, since our work focuses on values equal or below this threshold. Both are examples of additions that can be made to the model, that have not been considered in our work. Similarly to what we have done, one could study the impact of the government using the CEV model derived in this paper for its incentive policy, when the true generation process is different.

The type of project may also have a large impact on the modeling of the issues at hand (stimulation or prevention of investment). In our work, we have considered that the firm either invests or disinvests, with that decision being irreversible. We could extend the reasoning of our work by admitting that the firm may invest immediately with an option to disinvest after a certain period of time, for instance, or other compound options that would change the investment opportunity for the private firm and, hence, the incentive policy that should be put in place by the government.

Other smaller suggestions for future research include the consideration of other economic factors and dynamics, such as the issue of tax evasion (which would have an impact on the optimal tax policy), or that a more comprehensive numerical analysis be done, possibly in the form of application of the model to various countries in different stages of development and comparison of the results obtained.

Overall, our results show that if a stochastic pre-tax gross profit flows variable that follows a CEV process is treated as if it followed a GBM process (either mistakenly or for simplification), the incentive policies that are regarded as optimal would not achieve the desired outcome of the government. As thoroughly shown and explained, the model that we have presented for both stimulation of investment and prevention of disinvestment is useful as it permits the tractability of such a stochastic variable, by taking into consideration its true generation process. Hence, it can be a tool for the government in its analysis of the best incentive policies to use, especially in times of greater uncertainty.

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APPENDIX A

Derivation of the general solution for the value of investment/disinvestment opportunity when the project's pre-tax profit flows follow a GBM process

This derivation is well known in classical Real Options Valuation theory. Hence, it has been added to the dissertation as an Annex, to not overcomplicate the reading of the overall thesis. Nevertheless, it is presented here in a complete manner. We divide this demonstration in three parts, in which we will demonstrate the expressions (4.1) to (4.5) and expression (4.32) of this dissertation.

A.1. Expression (4.1)

We will begin our derivations going one step backward from expression (4.1). As it is stated in this model set up, it has been assumed that pre-tax profit flows follow a GBM process. This modelling assumption is useful since it guarantees three important properties: i) no negative values are allowed; ii) over the long run, an upwards trend exists; and iii) scaling the Brownian Motion increment by the volatility σ takes into account its possible different levels. So let us start with the following expression, to represent the GBM process:

$$\frac{dV\left(t\right)}{V\left(t\right)} = \mu^{\mathbb{P}} dt + \ \sigma dz^{\mathbb{P}}\left(t\right) \ \sim N\left(\mu dt, \sigma^{2} dt\right), \tag{A.1}$$

where d stands for the instantaneous change (i.e., over an infinitesimal time interval); $\frac{dV(t)}{V(t)}$ is the instantaneous return; $z^{\mathbb{P}}$ is a Brownian Motion ($dz^{\mathbb{P}}\left(t\right) = \ \varepsilon\sqrt{dt} \sim N(0,\ dt)$ with ε being the standard normal), and, finally, μ and σ are the project pre-tax profit flows' mean and volatility, respectively. Notice that we are still not under risk-neutral valuation, as the superscript $\mathbb{P}$ indicates that true probabilities are being used.

To move onto risk-neutral valuation, we must first define the market price of risk (ratio of expected excess return over the risk-free rate to the standard deviation). Hence, we can define it as:

$$\theta = \frac{\mu^{\mathbb{P}} + q - r}{\sigma}, \tag{A.2}$$

with r being the risk-free rate of return and q the dividend yield. We are now able to apply Girsanov's Theorem, which states that: if $Z^{\mathbb{P}}(t)$ is a Brownian Motion in the physical measure $\mathbb{P}$ (true probabilities), then $dZ^{\mathbb{Q}}\left(t\right) = dZ^{\mathbb{P}}\left(t\right) + \gamma dt$ is a Brownian Motion in an equivalent measure Q, for any $\gamma < \infty$. In this case, we define $\gamma = \ \theta$ and so:

$$
\begin{aligned}
&dV(t) = V(t)\mu^{\mathbb{P}}dt + \ V(t)\sigma(dz^{\mathbb{Q}}(t) - \theta dt)\\
\Leftrightarrow\ &dV(t) = V(t)\mu^{\mathbb{P}}dt + \ V(t)\sigma dz^{\mathbb{Q}}(t) - V(t)\sigma\theta dt\\
\Leftrightarrow\ &dV(t) = V(t)\left(\mu^{\mathbb{P}}dt - \ \sigma\theta dt\right) + V(t)\sigma dz^{\mathbb{Q}}(t)\\
\Leftrightarrow\ &dV(t) = V(t)\left(\mu^{\mathbb{P}}dt - \ \sigma\frac{\mu^{\mathbb{P}} + q - r}{\sigma}dt\right) + V(t)\sigma dz^{\mathbb{Q}}(t)\\
\Leftrightarrow\ &dV(t) = V(t)\left(\mu^{\mathbb{P}}dt - \mu^{\mathbb{P}}dt - qdt + rdt\right) + V(t)\sigma dz^{\mathbb{Q}}(t)\\
\Leftrightarrow\ &dV(t) = V(t)(r-q)dt + V(t)\sigma dz^{\mathbb{Q}}(t)\,.
\end{aligned}
\tag{A.3}
$$

Now, to reach the initial equation mentioned in the dissertation, which already considers risk-neutral terms, a few adjustments must be made to the expression to which we have arrived now. Since we are working in an infinite horizon, the expression becomes time independent. Moreover, the superscript $\mathbb{Q}$ disappears as it is made clear that from now on we will be working with the risk-neutral world. Lastly, it is defined that $\alpha = r - q$. Therefore, we arrive to expression (4.1):

$$dV = \alpha V dt + \sigma V dz. \tag{A.4}$$

Once again, α is the expected profit flows drift and σ is instantaneous volatility (both assumed to be constant), and dz is the increment of the Wiener process. Additionally, $\alpha < r$, and all entities are risk-neutral.

A.2. Expression (4.2)

Given this, we would like to find an expression for the value of the option to invest in the project which has its pre-tax profit flows governed as has been showed. Following notation given by Barbosa et al. (2016) in their paper, let us consider $H(V)$ to represent this value. Note that using risk-neutral valuation, then the expected rate of return on the project must equal the risk-free rate. That is:

$$\mathbb{E}\left[dH(V)\right] = rH(V)\,dt. \tag{A.5}$$

It is now important to remember that the GBM is an Itô Process. An Itô process is a process of the form:

$$dX_t = a(X_t, t)\,dt + b(X_t, t)\,dz_t. \tag{A.6}$$

Under GBM, and taking into account that we are working under time independence, it can be considered that $X_t = V, a(X_t, t) = \alpha V$ and $b(X_t, t) = \sigma V$. Then Itô's Lemma states that if X_t is an Itô process, then $f(X_t, t)$ is another Itô process given by:

$$df(X_t, t) = f_x(X_t, t)\,dX_t + f_t(X_t, t)\,dt + \frac{1}{2}f_{xx}(X_t, t)\,(dX_t)^2 \tag{A.7}$$

where $(dX_t)^2 = b(X_t, t)^2\,dt$ is the quadratic variation of X_t.

Taking this, we can expand the value of the investment opportunity using Ito's Lemma:

$$dH(V) = \frac{\partial H(V)}{\partial V} dV + \frac{1}{2}\frac{\partial^2 H(V)}{\partial V^2} dV^2. \tag{A.8}$$

To simplify the notation, we can substitute both $\frac{\partial H(V)}{\partial V} = H'(V)$ and $\frac{\partial^2 H(V)}{\partial V^2} = H''(V)$. Furthermore, we also know that $dV = \alpha V dt + \sigma V dz$. We only need to expand dV^2.

$$\begin{aligned} dV^2 &= (\alpha V dt + \sigma V dz)^2 \\ &= V^2\alpha^2 (dt)^2 + 2\sigma\alpha V^2 dtdz + \sigma^2 V^2 (dz)^2 . \end{aligned} \tag{A.9}$$

Let us now take into account the following points, extrapolated from Shreve (2004), considering that for the mathematical demonstrations we are taking a partition of the interval $[0, t]$ in n sub-intervals of t:

- $(dt)^2$ will tend to 0 faster than dt, and so $V^2\alpha^2 (dt)^2$ will tend to 0 faster than dt as well, and so this term will be removed from the expression above (to further explain, note that dt is an infinitesimal change in time, hence powers of dt, such as $(dt)^2$, will actually be quite irrelevant). It is nevertheless possible to prove this mathematically:

$$\begin{aligned} \int_0^t (dt)^2 &= \lim_{n\to\infty} \sum_{i=1}^{n} (t_i - t_{i-1})^2 \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{t}{n}\right)^2 \\ &= \lim_{n\to\infty} \frac{t^2}{n} = 0. \end{aligned} \tag{A.10}$$

- Remembering also that from the Brownian Motion definition we know that $dz = \varepsilon\sqrt{dt}\ \ N(0, dt)$, it is also possible to show that:

$$\begin{aligned} \int_0^t (dzdt) &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(z_t - z_{t_{i-1}}\right)(t_i - t_{i-1}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \varepsilon\sqrt{t_i - t_{i-1}}\,(t_i - t_{i-1}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \varepsilon\,(t_i - t_{i-1})^{\frac{3}{2}} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \varepsilon\left(\frac{t}{n}\right)^{\frac{3}{2}} \\ &= \lim_{n\to\infty} \frac{t^{\frac{3}{2}}}{n^{\frac{1}{2}}} \sum_{i=1}^{n} \frac{\varepsilon}{n} = 0. \end{aligned} \tag{A.11}$$

In differential notation, $(dzdt) = 0$ and so it is true that $2\sigma\alpha V^2 dzdt = 0$.

- Finally, seeing that $\varepsilon \sim N(0,1)$, then $Var(\varepsilon) = 1 \Leftrightarrow \mathbb{E}(\varepsilon^2) - [\mathbb{E}(\varepsilon)]^2 = 1 \Leftrightarrow \mathbb{E}(\varepsilon^2) = 1$ (since $\mathbb{E}(\varepsilon) = 0$), then it can also be shown that:

$$\begin{aligned}
\int_0^t (dz)^2 &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(z_{t_i} - z_{t_{i-1}}\right)^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\varepsilon\sqrt{t_i - t_{i-1}}\right)^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\varepsilon\sqrt{\frac{t}{n}}\right)^2 \\
&= \lim_{n\to\infty} t\sum_{i=1}^{n} \frac{{\varepsilon_i}^2}{n} \\
&= t\mathbb{E}\left[\varepsilon^2\right] = t.
\end{aligned} \tag{A.12}$$

In differential form, $(dz)^2 = dt$.

Taking these three points into account, then:

$$dV^2 = \sigma^2 V^2 dt. \tag{A.13}$$

Hence, considering the aforementioned arguments,

$$\begin{aligned}
dH(V) &= H'(V)(\alpha V dt + \sigma V dz) + \frac{1}{2}H''(V)(\sigma^2 V^2 dt) \\
\Leftrightarrow dH(V) &= H'(V)\alpha V dt + H'(V)\sigma V dz + \frac{1}{2}H''(V)\sigma^2 V^2 dt.
\end{aligned} \tag{A.14}$$

Taking the expectation:

$$\mathbb{E}[dH(V)] = \mathbb{E}[H'(V)\alpha V dt + H'(V)\sigma V dz + \frac{1}{2}H''(V)\sigma^2 V^2 dt]. \tag{A.15}$$

One more simplification can be made, considering that:

$$\begin{aligned}
\mathbb{E}[dz] &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(z_{t_i} - z_{t_{i-1}}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\varepsilon\sqrt{t_i - t_{i-1}}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \varepsilon\sqrt{\frac{t}{n}} \\
&= \lim_{n\to\infty} \frac{t^{\frac{1}{2}}}{n^{-\frac{1}{2}}} \sum_{i=1}^{n} \frac{\varepsilon_i}{n} \\
&= \frac{t^{\frac{1}{2}}}{n^{-\frac{1}{2}}}\mathbb{E}[\varepsilon] = 0.
\end{aligned} \tag{A.16}$$

And, so, we are left with:

$$\mathbb{E}\left[dH\left(V\right)\right] = H'\left(V\right)\alpha V dt + \frac{1}{2}H''\left(V\right)\sigma^2 V^2 dt. \tag{A.17}$$

Finally, we are ready to conclude this part of our demonstration:

$$\begin{aligned} & H'\left(V\right)\alpha V dt + \frac{1}{2}H''\left(V\right)\sigma^2 V^2 dt = rH\left(V\right)dt \\ \Leftrightarrow\ & \mathbb{E}\left[dH\left(V\right)\right] = rH\left(V\right)dt. \end{aligned} \tag{A.18}$$

Taking time independence and rearranging, we obtain the ODE referred to in expression (4.2) of this dissertation:

$$\begin{aligned} & H'\left(V\right)\alpha V + \frac{1}{2}H''\left(V\right)\sigma^2 V^2 = rH\left(V\right) \\ \Leftrightarrow\ & \frac{1}{2}\sigma^2 V^2 H''\left(V\right) + \alpha V H'\left(V\right) - rH\left(V\right) = 0. \end{aligned} \tag{A.19}$$

A.3. Expressions (4.3), (4.4), (4.5) and (4.32)

Now we will move on and show how to derive the general solution in expression (4.4), which must satisfy the above ODE. Notice that this ODE is linear in the dependent $H\left(V\right)$ and its derivatives, which means that its solution can be expressed as a linear combination of any two independent solutions (this comes from the superposition principle for the solutions of homogeneous linear differential equations, which claims that if $f_1, f_2, \ldots, f_k$ are all solutions to the differential equation, then for any constants $C_1, C_2, \ldots, C_k$, the function $C_1 f_1 + C_2 f_2 + \ldots + C_k f_k$ is also a solution to that equation) (Shreve, 2004).

If we try a solution of the form $H\left(V\right) = V^{\beta}$, it follows that $H'\left(V\right) = \beta V^{\beta-1}$ and $H''\left(V\right) = \beta\left(\beta-1\right)V^{\beta-2}$. Doing the substitution:

$$\begin{aligned} & \frac{1}{2}\sigma^2 V^2 H''\left(V\right) + \alpha V H'\left(V\right) - rH\left(V\right) = 0 \\ \Leftrightarrow\ & \frac{1}{2}\sigma^2 V^2 \beta\left(\beta-1\right)V^{\beta-2} + \alpha V \beta V^{\beta-1} - rV^{\beta} = 0 \\ \Leftrightarrow\ & \frac{1}{2}\sigma^2 \beta\left(\beta-1\right)V^{\beta} + \alpha\beta V^{\beta} - rV^{\beta} = 0 \\ \Leftrightarrow\ & V^{\beta}\left[\frac{1}{2}\sigma^2\beta\left(\beta-1\right) + \alpha\beta - r\right] = 0. \end{aligned} \tag{A.20}$$

$F\left(V\right) = V^{\beta}$ is a solution of the ODE if β is a root of the quadratic equation:

$$\mathcal{Q}\left(\beta\right) = \frac{1}{2}\sigma^2\beta\left(\beta-1\right) + \alpha\beta - r = 0. \tag{A.21}$$

We can multiply both sides of the equation by $2/\sigma^2$ and rearrange to obtain:

$$
\begin{aligned}
&\frac{1}{2}\sigma^2\frac{2}{\sigma^2}\beta\left(\beta-1\right)+\alpha\beta\frac{2}{\sigma^2}-r\frac{2}{\sigma^2}=0\\
\Leftrightarrow\ &\beta\left(\beta-1\right)+\beta\frac{2\alpha}{\sigma^2}-\frac{2r}{\sigma^2}=0\\
\Leftrightarrow\ &\beta^2-\beta+\beta\frac{2\alpha}{\sigma^2}-\frac{2r}{\sigma^2}=0\\
\Leftrightarrow\ &\beta^2-\beta\left(1-\frac{2\alpha}{\sigma^2}\right)-\frac{2r}{\sigma^2}=0.
\end{aligned}
\tag{A.22}
$$

Then we can consider that:

$$
\mathcal{Q}\left(\beta\right)=\beta^2-\beta\left(1-v\right)-w=0,
\tag{A.23}
$$

with $v=2\alpha/\sigma^2$ and $w=2r/\sigma^2$. This is clearly a quadratic equation, with a U-shape parabola form (in general, quadratic equations are of the form $y=ax^2+bx+c$, as is the case here; when $a>0$, then the parabola has a minimum point and opens upwards, which is the case with $\mathcal{Q}(\beta)$, where $a=1$).

The convergence condition is $w>v$, which implies that $q>0$, that is, a non-zero cost of waiting. This being so, $\mathcal{Q}\left(0\right)=-w<0$, because both r and σ^2 are positive; and $\mathcal{Q}\left(1\right)=1-1+v-w=-\left(w-v\right)<0$, given the convergence condition. Then, since $\mathcal{Q}'\left(\beta\right)=2\beta-\left(1-v\right)$ and so $\mathcal{Q}''\left(\beta\right)=2>0$, then it follows that $\mathcal{Q}(\beta)$ has two roots, one greater than one (β_1) and the other smaller than zero (β_2) (to further explain, this comes from knowing that the function has a U-shaped parabola form and that both $\mathcal{Q}\left(0\right)$ and $\mathcal{Q}\left(1\right)$ are negative values).

If the discriminant of the quadratic equation is larger than 0, then these are both real solutions. In general, for a quadratic function of the form $y=ax^2+bx+c$, the discriminant will be $\Delta=b^2-4ac$. Taking that here we have $a=1$, $b=-(1-v)$ and $c=-w$, then:

$$
\begin{aligned}
\Delta\ &=\left(-\left(1-v\right)\right)^2-4\left(-w\right)\\
&=1-2v+v^2+4w\\
&=1+v^2+2\left(2w-v\right).
\end{aligned}
\tag{A.24}
$$

Since all terms are positive ($2\left(2w-v\right)>0$ because of the convergence condition), then $\Delta>0$ and we know that both roots are real and equal to (remember that in a typical quadratic equation $y=ax^2+bx+c$, if $\Delta>0$, then the roots are given by $x=\frac{-b\pm\sqrt{\Delta}}{2a}$):

$$
\begin{aligned}
\beta_1 &= \frac{-\left(-\left(1-v\right)\right)+\sqrt{\Delta}}{2} \\
&= \frac{1-v+\sqrt{1+v^2+2\left(2w-v\right)}}{2} \\
&= \frac{1}{2}-\frac{v}{2}+\frac{\sqrt{1+v^2+2\left(2w-v\right)}}{2} \\
&= \frac{1}{2}-\frac{v}{2}+\frac{1}{2}\sqrt{1+v^2+4w-2v} \\
&= \frac{1}{2}-\frac{v}{2}+\frac{1}{2}\sqrt{\left(1-2v+v^2\right)+4w} \\
&= \frac{1}{2}-\frac{\alpha}{\sigma^2}+\frac{1}{2}\sqrt{1+\left(\frac{2\alpha}{\sigma^2}\right)^2+2\left(\frac{4r}{\sigma^2}-\frac{2\alpha}{\sigma^2}\right)} \qquad \text{(A.25)} \\
&= \frac{1}{2}-\frac{\alpha}{\sigma^2}+\sqrt{\frac{1}{4}}\sqrt{1+\left(\frac{2\alpha}{\sigma^2}\right)^2+2\left(\frac{4r}{\sigma^2}-\frac{2\alpha}{\sigma^2}\right)} \\
&= \frac{1}{2}-\frac{\alpha}{\sigma^2}+\sqrt{\frac{1}{4}+\frac{\alpha^2}{\sigma^4}+\left(\frac{2r}{\sigma^2}-\frac{\alpha}{\sigma^2}\right)} \\
&= \frac{1}{2}-\frac{\alpha}{\sigma^2}+\sqrt{\left(\frac{\alpha^2}{\sigma^4}-\frac{\alpha}{\sigma^2}+\frac{1}{4}\right)+\frac{2r}{\sigma^2}} \\
&= \frac{1}{2}-\frac{\alpha}{\sigma^2}+\sqrt{\left(\frac{\alpha}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}} > 1.
\end{aligned}
$$

And so we have arrived to expression (4.5) of the dissertation. Through a very similar process we also can get expression (4.32):

$$
\begin{aligned}
\beta_2 &= \frac{-\left(-\left(1-v\right)\right)-\sqrt{\Delta}}{2} \\
&= \frac{1}{2}-\frac{\alpha}{\sigma^2}-\sqrt{\left(\frac{\alpha}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}} < 0.
\end{aligned} \qquad \text{(A.26)}
$$

Then, as explained, the general solution of the ODE we have been working with can be written as stated in expression (4.3):

$$
H(V) = A_H V^{\beta_1} + B_H V^{\beta_2}, \qquad \text{(A.27)}
$$

where A_H and B_H are two constants to be determined from boundary conditions.

APPENDIX B

Demonstration of the expressions of the value of the investment opportunity and trigger to invest when the private firm undertakes the project

$$\begin{cases} P\left(V\right) = A_P V^{\beta_1} \\ P\left(\overline{V}_P\right) = \overline{V}_P(1-t_c) - \overline{X}_P \\ P\prime\left(\overline{V}_P\right) = 1-t_c. \end{cases} \tag{B.1}$$

Therefore, the constant A_P can be found using the value-matching condition:

$$\begin{aligned} A_P\overline{V}_P^{\beta_1} &= \overline{V}_P\left(1-t_c\right) - \overline{X}_P \\ \Leftrightarrow A_P &= \frac{\overline{V}_P\left(1-t_c\right)-\overline{X}_P}{\overline{V}_P^{\beta_1}}, \end{aligned} \tag{B.2}$$

which then means that:

$$\begin{aligned} P\left(V\right) &= \frac{\overline{V}_P\left(1-t_c\right)-\overline{X}_P}{\overline{V}_P}^{\beta_1} V^{\beta_1} \\ &= \left(\overline{V}_P\left(1-t_c\right)-\overline{X}_P\right)\left(\frac{V}{\overline{V}_P}\right)^{\beta_1}. \end{aligned} \tag{B.3}$$

This is valid when $V < \overline{V}_P$, since otherwise the firm invests (given that the investment trigger has been touched) and so the value of the investment opportunity is of course equal to the project after-tax profit flows minus the direct cost of investing. Hence,

$$P\left(V\right) = \begin{cases} \left(\overline{V}_P\left(1-t_c\right)-\overline{X}_P\right)\left(\frac{V}{\overline{V}_P}\right)^{\beta_1} \text{ for } V < \overline{V}_P \\ V\left(1-t_c\right) - \overline{X}_P \text{ for } V \geq \overline{V}_P. \end{cases} \tag{B.4}$$

Regarding the investment trigger, we turn to the smooth-pasting condition:

$$
\begin{aligned}
&\beta_1 A_P \overline{V}_P^{\,\beta_1-1} = 1 - t_c \\
\Leftrightarrow\ &\beta_1 \frac{\overline{V}_P\,(1-t_c) - \overline{X}_P}{\overline{V}_P^{\,\beta_1}} \overline{V}_P^{\,\beta_1-1} = 1 - t_c \\
\Leftrightarrow\ &\beta_1 (\overline{V}_P\,(1-t_c) - \overline{X}_P) \overline{V}_P^{\,-1} = 1 - t_c \\
\Leftrightarrow\ &\frac{\beta_1}{1-t_c} (\overline{V}_P\,(1-t_c) - \overline{X}_P) = \overline{V}_P \\
\Leftrightarrow\ &\beta_1 \overline{V}_P - \frac{\beta_1}{1-t_c} \overline{X}_P = \overline{V}_P \\
\Leftrightarrow\ &\overline{V}_P = \frac{\beta_1}{\beta_1 - 1} \frac{\overline{X}_P}{1-t_c}.
\end{aligned}
\tag{B.5}
$$

APPENDIX C

Demonstration of the expressions of the value of the investment opportunity and trigger to invest when the public sector undertakes the project

$$\begin{cases} G(V) = A_G V_g^{\beta_1} \\ G\left(\overline{V}_G\right) = \overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G \\ G'\left(\overline{V}_G\right) = 1. \end{cases} \tag{C.1}$$

Notice that in boundary condition I we have V_g instead of V, since we account for public inefficiency in extracting profits from the project; V_g is the relevant project value for the government.

Therefore, the constant A_G can be found using the value-matching condition:

$$\begin{aligned} A_G \overline{V}_G^{\beta_1} &= \overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G \\ \Leftrightarrow A_G &= \frac{\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G}{\overline{V}_G^{\beta_1}}. \end{aligned} \tag{C.2}$$

Hence,

$$\begin{aligned} G(V) &= \frac{\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G}{\overline{V}_G^{\beta_1}} V_g^{\beta_1} \\ \Leftrightarrow G(V) &= \left(\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G\right) \left(\frac{V_g}{\overline{V}_G}\right)^{\beta_1}. \end{aligned} \tag{C.3}$$

This is valid when $V_g < \overline{V}_G$, since otherwise the public sector invests (since the investment trigger has been touched) and so the value of the investment opportunity is of course equal to project gains minus its direct cost. That is:

$$G(V) = \begin{cases} \left(\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G\right) \left(\frac{V_g}{\overline{V}_G}\right)^{\beta_1} & \text{for } V_g < \overline{V}_G \\ V_g - \overline{X}_G + t_n \lambda_G \overline{X}_G & \text{for } V_g \geq \overline{V}_G. \end{cases} \tag{C.4}$$

When it comes to the investment trigger, we use the smooth-pasting condition:

$$
\begin{aligned}
&\beta_1 A_G \overline{V}_G^{\beta_1 - 1} = 1 \\
\Leftrightarrow\ &\beta_1 \frac{\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G}{\overline{V}_G^{\beta_1}} \overline{V}_G^{\beta_1 - 1} = 1 \\
\Leftrightarrow\ &\beta_1 (\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G) = \overline{V}_G \\
\Leftrightarrow\ &\beta_1 (-\overline{X}_G + t_n \lambda_G \overline{X}_G) = \overline{V}_G - \beta_1 \overline{V}_G \\
\Leftrightarrow\ &\frac{\beta_1}{1 - \beta_1} \overline{X}_G \left(-1 + t_n \lambda_G\right) = \overline{V}_G \\
\Leftrightarrow\ &\overline{V}_G = \frac{\beta_1}{\beta_1 - 1} (1 - t_n \lambda_G) \overline{X}_G.
\end{aligned}
\tag{C.5}
$$

APPENDIX D

Demonstration of the expression which resumes the comparison between a subsidy and a tax policy to stimulate investment

Let us first note that:

$$
\begin{aligned}
\Pi_{GS}\left(S=S_{opt}\right) &= t_c V + t_n \lambda_P \overline{X}_P - \left(\overline{X}_P - \frac{\beta_1 - 1}{\beta_1} V\left(1-t_c\right)\right) \\
&= t_c V + t_n \lambda_P \overline{X}_P - \overline{X}_P + \frac{\beta_1 - 1}{\beta_1} V\left(1-t_c\right) \\
&= t_c V + \overline{X}_P (t_n \lambda_P - 1) + \frac{\beta_1 - 1}{\beta_1} V\left(1-t_c\right) \\
&= V\left(t_c\left(1 - \frac{\beta_1 - 1}{\beta_1}\right) + \frac{\beta_1 - 1}{\beta_1}\right) + \overline{X}_P (t_n \lambda_P - 1)
\end{aligned}
\tag{D.1}
$$

and:

$$
\begin{aligned}
\Pi_{GS}(t_c = \bar{t}_{c_{opt}}; S=0) &= \left(1 - \frac{\beta_1}{\beta_1 - 1}\frac{\overline{X}_P}{V}\right) V + t_n \lambda_P \overline{X}_P \\
&= V + \left(t_n \lambda_P - \frac{\beta_1}{\beta_1 - 1}\right)\overline{X}_P .
\end{aligned}
\tag{D.2}
$$

Let us also assume that $\Pi_{GS}\left(S=S_{opt}\right) \leq \Pi_{GS}(t_c = \bar{t}_{c_{opt}}; S=0)$ and simplify the inequality, in order to verify that it can be true:

$$
\begin{aligned}
& V\left(t_c\left(1 - \frac{\beta_1 - 1}{\beta_1}\right) + \frac{\beta_1 - 1}{\beta_1}\right) + \overline{X}_P (t_n \lambda_P - 1) \leq V + \left(t_n \lambda_P - \frac{\beta_1}{\beta_1 - 1}\right)\overline{X}_P \\
\Leftrightarrow\ & V\left(t_c\left(1 - \frac{\beta_1 - 1}{\beta_1}\right) + \frac{\beta_1 - 1}{\beta_1} - 1\right) - \overline{X}_P \leq -\frac{\beta_1}{\beta_1 - 1}\overline{X}_P \\
\Leftrightarrow\ & V t_c\left(1 - \frac{\beta_1 - 1}{\beta_1}\right) + V\left(\frac{\beta_1 - 1}{\beta_1} - 1\right) \leq \left(1 - \frac{\beta_1}{\beta_1 - 1}\right)\overline{X}_P \\
\Leftrightarrow\ & V t_c\left(1 - \frac{\beta_1 - 1}{\beta_1}\right) - V\left(1 - \frac{\beta_1 - 1}{\beta_1}\right) \leq \left(1 - \frac{\beta_1}{\beta_1 - 1}\right)\overline{X}_P .
\end{aligned}
\tag{D.3}
$$

Noting that $\beta_1 > 1$, then it must be that $\frac{\beta_1 - 1}{\beta_1} > 1$ and, therefore, $\left(1 - \frac{\beta_1 - 1}{\beta_1}\right) > 1$, meaning that we can divide both sides of the equation by this term, being left with:

$$
\begin{aligned}
Vt_c - V &\leq \frac{\left(1 - \frac{\beta_1}{\beta_1 - 1}\right)\overline{X}_P}{\left(1 - \frac{\beta_1 - 1}{\beta_1}\right)} \\
\Leftrightarrow\ V(t_c - 1) &\leq \frac{\left(-\frac{1}{\beta_1 - 1}\right)}{\left(\frac{1}{\beta_1}\right)}\overline{X}_P \\
\Leftrightarrow\ V(t_c - 1) &\leq \frac{\beta_1}{1 - \beta_1}\overline{X}_P \\
\Leftrightarrow\ V &\geq \frac{\beta_1}{\beta_1 - 1}\frac{\overline{X}_P}{1 - t_c}.
\end{aligned}
\tag{D.4}
$$

Given that t_c is a corporate tax rate, then $0 \leq t_c \leq 1$, which would mean that $(t_c - 1) \leq 0$, which is why the direction of the inequation changes when we move this term to the right side of the inequality.

Now remember that $\overline{V}_P = \frac{\beta_1}{\beta_1 - 1}\frac{\overline{X}_P}{1 - t_c}$. Then see that if we substitute the latter expression into the inequality we arrive to:

$$
V \geq \overline{V}_P. \tag{D.5}
$$

This means that the inequality $\Pi_{GS}\,(S = S_{opt}) \leq \Pi_{GS}(t_c = \bar{t}_{c_{opt}}; S = 0)$ is true for whenever $V \geq \overline{V}_P$. Therefore, in the context of stimulating investment, meaning when $V < \overline{V}_P$, its alternative must be true: $\Pi_{GS}\,(S = S_{opt}) > \Pi_{GS}\left(t_c = \bar{t}_{c_{opt}}; S = 0\right)$.

APPENDIX E

Demonstration of the expressions of the value of the disinvestment opportunity and trigger to disinvest

$$\begin{cases} P\left(V\right) = B_P V^{\beta_2} \\ P\left(\underline{V}_P\right) = \underline{X} - \underline{V}_P\left(1-t_c\right) \\ P'\left(\underline{V}_P\right) = -(1-t_c). \end{cases} \tag{E.1}$$

Using the value matching condition:

$$\begin{aligned} & B_P \underline{V}_P{}^{\beta_2} = \underline{X} - \underline{V}_P\left(1-t_c\right) \\ \Leftrightarrow \;\; & B_P = \frac{\underline{X} - \underline{V}_P\left(1-t_c\right)}{\underline{V}_P{}^{\beta_2}}. \end{aligned} \tag{E.2}$$

So,

$$\begin{aligned} & P\left(V\right) = \frac{\underline{X} - \underline{V}_P\left(1-t_c\right)}{\underline{V}_P{}^{\beta_2}} V^{\beta_2} \\ \Leftrightarrow \;\; & P\left(V\right) = \left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right)\left(\frac{V}{\underline{V}_P}\right)^{\beta_2}, \end{aligned} \tag{E.3}$$

which is only valid when $V > \underline{V}_P$. If this is not true, then the disinvestment trigger has been touched and the option exercised. Hence, the value of the option to disinvest can be written as:

$$P\left(V\right) = \begin{cases} \left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right)\left(\frac{V}{\underline{V}_P}\right)^{\beta_2} & \text{for} \;\; V > \underline{V}_P \\ \underline{X} - \underline{V}_P\left(1-t_c\right) \;\; \text{for} \;\; V \leq \underline{V}_P. \end{cases} \tag{E.4}$$

Now using the smooth-pasting condition to derive the expression for $\underline{V}_P$, we do:

$$\begin{aligned} & \beta_2 B_P \underline{V}_P{}^{\beta_2 - 1} = -(1-t_c) \\ \Leftrightarrow \; & \beta_2 \frac{\underline{X} - \underline{V}_P\left(1-t_c\right)}{\underline{V}_P{}^{\beta_2}} V^{\beta_2 - 1} = t_c - 1 \\ \Leftrightarrow \; & \beta_2\left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right)\underline{V}_P{}^{-1} = t_c - 1 \\ \Leftrightarrow \; & \frac{\beta_2}{t_c - 1}\underline{X} - \frac{\beta_2}{t_c - 1}\underline{V}_P\left(1-t_c\right) = \underline{V}_P \\ \Leftrightarrow \; & \underline{V}_P = \frac{\beta_2}{1-\beta_2}\frac{\underline{X}}{t_c - 1}. \end{aligned} \tag{E.5}$$

APPENDIX F

Derivation of the expression for the first derivative of the value of the investment opportunity under CEV, when $r = q$

$$
\begin{aligned}
P'(V) &= \frac{\partial}{\partial V}\left[C_P V^{\frac{1}{2}} I_s(z(V))\right] \\
&= \frac{\partial}{\partial V}\left[C_P V^{\frac{1}{2}}\right] I_s(z(V)) + C_P V^{\frac{1}{2}} \frac{\partial}{\partial V}\left[I_s(z(V))\right] \\
&= \frac{C_P}{2} V^{-\frac{1}{2}} I_s\left(z(V)\right) + C_P V^{\frac{1}{2}} \frac{\partial}{\partial z}\left[I_s\left(z(V)\right)\right] \frac{\partial z\left(V\right)}{\partial V}.
\end{aligned}
\tag{F.1}
$$

Remember that:

$$
z\left(V\right) = \frac{2\sqrt{2r}}{\delta|\beta-2|} V^{1-\frac{\beta}{2}}. \tag{F.2}
$$

Being so,

$$
\frac{\partial z\left(V\right)}{\partial V} = \left(1 - \frac{\beta}{2}\right) \frac{2\sqrt{2r}}{\delta\left|\beta-2\right|} V^{-\frac{\beta}{2}}. \tag{F.3}
$$

Since $\beta < 2$, then $\frac{2-\beta}{|\beta-2|} = 1$. Hence,

$$
\frac{\partial z\left(V\right)}{\partial V} = \frac{\sqrt{2r}}{\delta} V^{-\frac{\beta}{2}}. \tag{F.4}
$$

Additionally, we know that:

$$
\frac{\partial I_s\left(z(V)\right)}{\partial z} = \frac{s}{z} I_s\left(z(V)\right) + I_{s+1}\left(z(V)\right). \tag{F.5}
$$

Doing the appropriate substitutions:

$$
P'\left(V\right) = \frac{C_P}{2} V^{-\frac{1}{2}} I_s\left(z(V)\right) + C_P V^{\frac{1}{2}} \frac{\sqrt{2r}}{\delta} V^{-\frac{\beta}{2}} \left[\frac{\frac{1}{|\beta-2|}}{\frac{2\sqrt{2r}}{\delta|\beta-2|V^{1-\frac{\beta}{2}}}} I_s\left(z(V)\right) + I_{s+1}\left(z(V)\right)\right]. \tag{F.6}
$$

Since $\beta < 2$, then:

$$
\begin{aligned}
P'(V) &= \frac{C_P}{2}V^{-\frac{1}{2}}I_s\left(z(V)\right) + C_P V^{\frac{1-\beta}{2}}\frac{\sqrt{2r}}{\delta}\left[\frac{\delta}{2\sqrt{2r}V^{1-\frac{\beta}{2}}}I_s\left(z(V)\right) + I_{s+1}\left(z(V)\right)\right] \\
&= \frac{C_P}{2}V^{-\frac{1}{2}}I_s\left(z(V)\right) + \frac{C_P}{2}V^{-\frac{1}{2}}I_s\left(z(V)\right) + C_P V^{\frac{1-\beta}{2}}\frac{\sqrt{2r}}{\delta}I_{s+1}\left(z(V)\right) \\
&= C_P V^{-\frac{1}{2}}I_s\left(z(V)\right) + C_P V^{\frac{1-\beta}{2}}\frac{\sqrt{2r}}{\delta}I_{s+1}\left(z(V)\right).
\end{aligned}
\tag{F.7}
$$

APPENDIX G

Derivation of the expression used to numerically find the private firm's investment trigger value under CEV, when $r = q$

Taking the system of equations:

$$\begin{cases} C_P \overline{V}_P^{\frac{1}{2}} I_s\left(z(\overline{V}_P)\right) = \overline{V}_P \ (1-t_c) - \overline{X}_P \\ C_P \overline{V}_P^{-\frac{1}{2}} I_s\left(z(\overline{V}_P)\right) + C_P \overline{V}_P^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(\overline{V}_P)\right) = 1 - t_c. \end{cases} \tag{G.1}$$

It is straightforward that from the first equation in the system we can derive the expression for C_P:

$$\begin{aligned} & C_P \overline{V}_P^{\frac{1}{2}} I_s\left(z(\overline{V}_P)\right) = \overline{V}_P \ (1-t_c) - \overline{X}_P \\ \Leftrightarrow \ & C_P = \frac{\overline{V}_P\,(1-t_c) - \overline{X}_P}{\overline{V}_P^{\frac{1}{2}} I_s\left(z(\overline{V}_P)\right)} \\ \Leftrightarrow \ & C_P = \frac{1}{I_s\left(z(\overline{V}_P)\right)} \overline{V}_P^{-\frac{1}{2}} (\overline{V}_P\,(1-t_c) - \overline{X}_P). \end{aligned} \tag{G.2}$$

We can then substitute C in the second equation of the system, to obtain an expression that allows us to solve for $V^*_{P_{CEV\_inv}}$ numerically.

$$\begin{aligned} & C_P \overline{V}_P^{-\frac{1}{2}} I_s\left(z(\overline{V}_P)\right) + C_P \overline{V}_P^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(\overline{V}_P)\right) = 1 - t_c \\ \Leftrightarrow \ & \frac{1}{I_s\left(z(\overline{V}_P)\right)} \overline{V}_P^{-\frac{1}{2}} (\overline{V}_P\,(1-t_c) - \overline{X}_P) \overline{V}_P^{-\frac{1}{2}} I_s\left(z(\overline{V}_P)\right) \\ & \quad + \frac{1}{I_s\left(z(\overline{V}_P)\right)} \overline{V}_P^{-\frac{1}{2}} (\overline{V}_P\,(1-t_c) - \overline{X}_P) \overline{V}_P^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(\overline{V}_P)\right) = 1 - t_c \\ \Leftrightarrow \ & (1-t_c) - \frac{\overline{X}_P}{\overline{V}_P} + \overline{V}_P^{-\frac{\beta}{2}} \left(\overline{V}_P\,(1-t_c) - \overline{X}_P\right) \frac{\sqrt{2r}}{\delta} \frac{I_{s+1}\left(z(\overline{V}_P)\right)}{I_s\left(z(\overline{V}_P)\right)} - (1-t_c) = 0 \\ \Leftrightarrow \ & -\overline{X}_P + \overline{V}_P^{\frac{2-\beta}{2}} \left(\overline{V}_P\,(1-t_c) - \overline{X}_P\right) \frac{\sqrt{2r}}{\delta} \frac{I_{s+1}\left(z(\overline{V}_P)\right)}{I_s\left(z(\overline{V}_P)\right)} = 0. \end{aligned} \tag{G.3}$$

APPENDIX H

Derivation of the expression for the first derivative of the value of the investment opportunity under CEV, when $r \neq q$

The derivation is as follows, with the derivative of $M(a, b, x(V))$ taken from Abramowitz and Stegun (1972), equation 13.4.8.:

$$
\begin{aligned}
P'(V) &= \frac{\partial}{\partial V}\left[C_P e^{\frac{(\varepsilon-1)x(V)}{2}} yVM(a,b,x(V))\right] \\
&= \frac{\partial}{\partial V}\left[C_P e^{\frac{(\varepsilon-1)x(V)}{2}}\right] yVM(a,b,x(V)) + C_P e^{\frac{(\varepsilon-1)x(V)}{2}} \frac{\partial}{\partial V}[yV] M(a,b,x(V)) \\
&\quad + C_P e^{\frac{(\varepsilon-1)x(V)}{2}} yV \frac{\partial}{\partial x}[M(a,b,x(V))] x'(V) \\
&= C_P e^{\frac{(\varepsilon-1)x(V)}{2}} \frac{(\varepsilon-1)}{2} x'(V) yVM(a,b,x(V)) + C_P e^{\frac{(\varepsilon-1)x(V)}{2}} yM(a,b,x(V)) \\
&\quad + C_P e^{\frac{(\varepsilon-1)x(V)}{2}} yV \frac{a}{b} M(a+1,b+1,x(V)) x'(V) \\
&= C_P e^{\frac{(\varepsilon-1)x(V)}{2}} yVM(a,b,x(V)) \frac{(\varepsilon-1)}{2} x'(V) \frac{V}{V} + C_P e^{\frac{(\varepsilon-1)x(V)}{2}} yVM(a,b,x(V)) \frac{1}{V} \\
&\quad + C_P e^{\frac{(\varepsilon-1)x(V)}{2}} yVM(a,b,x(V)) \frac{a}{b} \frac{M(a+1,b+1,x(V))}{M(a,b,x(V))} x'(V) \\
&= P(V) \times \frac{(\varepsilon-1)}{2} x'(V) + P(V) \times \frac{1}{V} + P(V) \times \frac{a}{b} \frac{M(a+1,b+1,x(V))}{M(a,b,x(V))} x'(V) \\
&= P(V) \times \left[\frac{(\varepsilon-1)}{2} x'(V) + \frac{1}{V} + \frac{a}{b} \frac{M(a+1,b+1,x(V))}{M(a,b,x(V))} x'(V)\right] \\
&= P(V) \times \left[V^{-1} + x'(V)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b} \frac{M(a+1,b+1,x(V))}{M(a,b,x(V))}\right]\right].
\end{aligned} \tag{H.1}
$$

APPENDIX I

Derivation of the expression used to numerically find the private firm's investment trigger value under CEV, when $r \neq q$

We will consider the system of equations:

$$\begin{cases} C_P e^{\frac{(\varepsilon-1)x(\overline{V}_P)}{2}} y\overline{V}_P M\left(a,b,x(\overline{V}_P)\right) = \overline{V}_P \ (1-t_c) - \overline{X}_P \\ P\left(\overline{V}_P\right) \times \left[\overline{V}_P^{\,-1} + x'(\overline{V}_P)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_P)\right)}{M\left(a,b,x(\overline{V}_P)\right)}\right]\right] = 1-t_c. \end{cases} \tag{I.1}$$

Taking the second equation of the system and the fact that:

$$\begin{aligned} & C_P e^{\frac{(\varepsilon-1)x(\overline{V}_P)}{2}} y\overline{V}_P M\left(a,b,x(\overline{V}_P)\right) = \overline{V}_P \ (1-t_c) - \overline{X}_P \\ \Leftrightarrow\ & P\left(\overline{V}_P\right) = \overline{V}_P\,(1-t_c) - \overline{X}_P, \end{aligned} \tag{I.2}$$

then,

$$\begin{aligned} & \overline{V}_P^{\,-1} + x'(\overline{V}_P)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_P)\right)}{M\left(a,b,x(\overline{V}_P)\right)}\right] = \frac{1-t_c}{\overline{V}_P\,(1-t_c)-\overline{X}_P} \\ \Leftrightarrow\ & -\frac{1-t_c}{\overline{V}_P\,(1-t_c)-\overline{X}_P}\frac{1}{\overline{V}_P} \\ & \qquad + x'(\overline{V}_P)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_P)\right)}{M\left(a,b,x(\overline{V}_P)\right)}\right] = 0 \\ \Leftrightarrow\ & -\overline{V}_P\,(1-t_c) + \overline{V}_P\,(1-t_c) - \overline{X}_P + \overline{V}_P(\overline{V}_P\,(1-t_c)-\overline{X}_P)x'(\overline{V}_P) \\ & \qquad \times\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_P)\right)}{M\left(a,b,x(\overline{V}_P)\right)}\right] = 0 \\ \Leftrightarrow\ & -\overline{X}_P + \overline{V}_P(\overline{V}_P\,(1-t_c)-\overline{X}_P)x'(\overline{V}_P)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_P)\right)}{M\left(a,b,x(\overline{V}_P)\right)}\right] = 0. \end{aligned} \tag{I.3}$$

Seeing that:

$$
\begin{aligned}
x'(V) &= \frac{\partial}{\partial V}\left[\frac{2\,|r-q|}{\delta^2\,|\beta-2|}V^{2-\beta}\right] \\
&= (2-\beta)\frac{2\,|r-q|}{\delta^2\,|\beta-2|}V^{1-\beta} \\
&= \frac{2-\beta}{V}x(V),
\end{aligned}
\tag{I.4}
$$

then it follows that:

$$
\begin{aligned}
& -\overline{X}_P+\overline{V}_P(\overline{V}_P\,(1-t_c)-\overline{X}_P)\frac{2-\beta}{\overline{V}_P}x(\overline{V}_P)\left[\frac{(\varepsilon-1)}{2}+\frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_P)\right)}{M\left(a,b,x(\overline{V}_P)\right)}\right]=0 \\
\Leftrightarrow\quad & -\overline{X}_P+(\overline{V}_P\,(1-t_c)-\overline{X}_P)(2-\beta)x(\overline{V}_P)\left[\frac{(\varepsilon-1)}{2}+\frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_P)\right)}{M\left(a,b,x(\overline{V}_P)\right)}\right]=0.
\end{aligned}
\tag{I.5}
$$

Furthermore, from the first equation on the system, and in a quite straightforward manner:

$$
C_P=\left(\overline{V}_P\,(1-t_c)-\overline{X}_P\right)\times\overline{V}_P^{\;-1}\times\frac{1}{e^{\frac{(\varepsilon-1)x(\overline{V}_P)}{2}}yM\left(a,b,x(\overline{V}_P)\right)}. \tag{I.6}
$$

APPENDIX J

Derivation of the expression used to numerically find the public sector's investment trigger value under CEV, when $r = q$

$$\begin{cases} C_G \overline{V}_G^{\frac{1}{2}} I_s\left(z(\overline{V}_G)\right) = \overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G \\ C_G \overline{V}_G^{-\frac{1}{2}} I_s\left(z(\overline{V}_G)\right) + C_G \overline{V}_G^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(\overline{V}_G)\right) = 1. \end{cases} \tag{J.1}$$

Then, from the first equation:

$$C_G = \left(\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G\right) \overline{V}_G^{-\frac{1}{2}} \frac{1}{I_s\left(z(\overline{V}_G)\right)}. \tag{J.2}$$

Substituting C_G in the second expression we are able to obtain the expression we need to solve $\overline{V}_G$ numerically:

$$\begin{aligned} & C_G \overline{V}_G^{-\frac{1}{2}} I_s\left(z(\overline{V}_G)\right) + C_G \overline{V}_G^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(\overline{V}_G)\right) = 1 \\ \Leftrightarrow\ & \left(\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G\right) \overline{V}_G^{-\frac{1}{2}} \frac{1}{I_s\left(z(\overline{V}_G)\right)} \overline{V}_G^{-\frac{1}{2}} I_s\left(z(\overline{V}_G)\right) \\ & + \left(\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G\right) \overline{V}_G^{-\frac{1}{2}} \frac{1}{I_s\left(z(\overline{V}_G)\right)} \overline{V}_G^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} I_{s+1}\left(z(\overline{V}_G)\right) = 1 \\ \Leftrightarrow\ & \left(-\overline{X}_G + t_n \lambda_G \overline{X}_G\right) + \overline{V}_G^{\frac{2-\beta}{2}} \left(\overline{V}_G - \overline{X}_G + t_n \lambda_G \overline{X}_G\right) \frac{\sqrt{2r}}{\delta} \frac{I_{s+1}\left(z(\overline{V}_G)\right)}{I_s\left(z(\overline{V}_G)\right)} = 0. \end{aligned} \tag{J.3}$$

APPENDIX K

Derivation of the expression used to numerically find the public sector's investment trigger value under CEV, when $r \neq q$

From the system of equations:

$$\begin{cases} C_G e^{\frac{(\varepsilon-1)x(\overline{V}_G)}{2}} y\overline{V}_G M\left(a,b,x(\overline{V}_G)\right) = \overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G \\ G\left(\overline{V}_G\right) \times \left[\overline{V}_G^{\,-1} + x'(\overline{V}_G)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_G)\right)}{M\left(a,b,x(\overline{V}_G)\right)}\right]\right] = 1. \end{cases} \tag{K.1}$$

and following the same reasoning as in Appendix I:

$$\begin{aligned} & C_G e^{\frac{(\varepsilon-1)x(\overline{V}_G)}{2}} y\overline{V}_G M\left(a,b,x(\overline{V}_G)\right) = \overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G \\ \Leftrightarrow\ & G\left(\overline{V}_G\right) = \overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G, \end{aligned} \tag{K.2}$$

so,

$$\begin{aligned} & \frac{1}{\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G} + \frac{1}{\overline{V}_G} \\ & \qquad + x'(\overline{V}_G)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_G)\right)}{M\left(a,b,x(\overline{V}_G)\right)}\right] = 0 \\ \Leftrightarrow\ & -\overline{V}_G + \overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G + \overline{V}_G\left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right) \\ & \qquad \times x'(\overline{V}_G)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_G)\right)}{M\left(a,b,x(\overline{V}_G)\right)}\right] = 0 \\ \Leftrightarrow\ & -\overline{X}_G + t_n\lambda_G\overline{X}_G + \overline{V}_G\left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right)\frac{2-\beta}{\overline{V}_G} \\ & \qquad \times x(\overline{V}_G)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_G)\right)}{M\left(a,b,x(\overline{V}_G)\right)}\right] = 0 \\ \Leftrightarrow\ & -\overline{X}_G + t_n\lambda_G\overline{X}_G + \left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right)(2-\beta) \\ & \qquad \times x(\overline{V}_G)\left[\frac{(\varepsilon-1)}{2} + \frac{a}{b}\frac{M\left(a+1,b+1,x(\overline{V}_G)\right)}{M\left(a,b,x(\overline{V}_G)\right)}\right] = 0 \end{aligned} \tag{K.3}$$

Finally, it is clear that:

$$C_G = \left(\overline{V}_G - \overline{X}_G + t_n\lambda_G\overline{X}_G\right) \times \overline{V}_G^{\,-1} \times \frac{1}{e^{\frac{(\varepsilon-1)x(\overline{V}_G)}{2}} yM\left(a,b,x(\overline{V}_G)\right)}. \tag{K.4}$$

APPENDIX L

Derivation of the expression used to numerically find the private firm's disinvestment trigger value under CEV, when $r = q$

$$\begin{cases} D_P \underline{V}_P^{\frac{1}{2}} K_s\left(z(\underline{V}_P)\right) = \underline{X} - \underline{V}_P\left(1-t_c\right) \\ D_P \underline{V}_P^{-\frac{1}{2}} K_s\left(z(\underline{V}_P)\right) - D_P \underline{V}_P^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} K_{s+1}\left(z(\underline{V}_P)\right) = -\left(1-t_c\right). \end{cases} \tag{L.1}$$

From the first equation it is clear that:

$$\begin{aligned} D_P &= \frac{\underline{X} - \underline{V}_P\left(1-t_c\right)}{\underline{V}_P^{\frac{1}{2}} K_s\left(z(\underline{V}_P)\right)} \\ \Leftrightarrow D_P &= \frac{1}{K_s\left(z(\underline{V}_P)\right)} \underline{V}_P^{-\frac{1}{2}}\left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right). \end{aligned} \tag{L.2}$$

Given this and looking at the second equation in the system, then:

$$\begin{aligned} &\frac{1}{K_s\left(z(\underline{V}_P)\right)} \underline{V}_P^{-\frac{1}{2}}\left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right) \underline{V}_P^{-\frac{1}{2}} K_s\left(z(\underline{V}_P)\right) \\ &\quad - \frac{1}{K_s\left(z(\underline{V}_P)\right)} \underline{V}_P^{-\frac{1}{2}}\left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right) \\ &\quad \times \underline{V}_P^{\frac{1-\beta}{2}} \frac{\sqrt{2r}}{\delta} K_{s+1}\left(z(\underline{V}_P)\right) = -\left(1-t_c\right) \\ \Leftrightarrow\ &\frac{\underline{X}}{\underline{V}_P} - \left(1-t_c\right) - \underline{V}_P^{-\frac{\beta}{2}}\left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right) \\ &\quad \times \frac{\sqrt{2r}}{\delta} \frac{K_{s+1}\left(z(\underline{V}_P)\right)}{K_s\left(z(\underline{V}_P)\right)} + \left(1-t_c\right) = 0 \\ \Leftrightarrow\ &\underline{X} - \underline{V}_P^{\frac{2-\beta}{2}}\left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right) \frac{\sqrt{2r}}{\delta} \frac{K_{s+1}\left(z(\underline{V}_P)\right)}{K_s\left(z(\underline{V}_P)\right)} = 0. \end{aligned} \tag{L.3}$$

APPENDIX M

Derivation of the expression used to numerically find the private firm's disinvestment trigger value under CEV, when $r \neq q$

$$\begin{cases} D_P e^{\frac{(\varepsilon-1)x(\underline{V}_P)}{2}} y \underline{V}_P U\left(a,b,x(\underline{V}_P)\right) = \underline{X} - \underline{V}_P\left(1-t_c\right) \\ P\left(\underline{V}_P\right) \times \left[\underline{V}_P{}^{-1} + x'(\underline{V}_P)\left[\frac{(\varepsilon-1)}{2} - a\frac{U(a+1,b+1,x(\underline{V}_P))}{U(a,b,x(\underline{V}_P))}\right]\right] = -\left(1-t_c\right) \end{cases} \tag{M.1}$$

Seeing that:

$$\begin{aligned} D_P e^{\frac{(\varepsilon-1)x(\underline{V}_P)}{2}} y \underline{V}_P U\left(a,b,x(\underline{V}_P)\right) &= \underline{X} - \underline{V}_P\left(1-t_c\right) \\ \Leftrightarrow P\left(\underline{V}_P\right) &= \underline{X} - \underline{V}_P\left(1-t_c\right). \end{aligned} \tag{M.2}$$

Then, from the second equation:

$$\begin{aligned} &\underline{V}_P{}^{-1} + x'\left(\underline{V}_P\right)\left[\frac{(\varepsilon-1)}{2} - a\frac{U\left(a+1,b+1,x(\underline{V}_P)\right)}{U\left(a,b,x(\underline{V}_P)\right)}\right] = -\frac{\left(1-t_c\right)}{\underline{X} - \underline{V}_P\left(1-t_c\right)} \\ &\Leftrightarrow \frac{\left(1-t_c\right)}{\underline{X} - \underline{V}_P\left(1-t_c\right)} + \frac{1}{\underline{V}_P} \\ &\qquad + x'\left(\underline{V}_P\right)\left[\frac{(\varepsilon-1)}{2} - a\frac{U\left(a+1,b+1,x(\underline{V}_P)\right)}{U\left(a,b,x(\underline{V}_P)\right)}\right] = 0 \\ &\Leftrightarrow \underline{V}_P\left(1-t_c\right) + \underline{X} - \underline{V}_P\left(1-t_c\right) + \underline{V}_P\left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right) \\ &\qquad x'\left(\underline{V}_P\right)\left[\frac{(\varepsilon-1)}{2} - a\frac{U\left(a+1,b+1,x(\underline{V}_P)\right)}{U\left(a,b,x(\underline{V}_P)\right)}\right] = 0 \\ &\Leftrightarrow \underline{X} + \left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right)\left(2-\beta\right) \\ &\qquad x\left(\underline{V}_P\right)\left[\frac{(\varepsilon-1)}{2} - a\frac{U\left(a+1,b+1,x(\underline{V}_P)\right)}{U\left(a,b,x(\underline{V}_P)\right)}\right] = 0. \end{aligned} \tag{M.3}$$

Finally, from the first equation:

$$D_P = \left(\underline{X} - \underline{V}_P\left(1-t_c\right)\right)\underline{V}_P{}^{-1}\frac{1}{e^{\frac{(\varepsilon-1)x(\underline{V}_P)}{2}} y U\left(a,b,x(\underline{V}_P)\right)}. \tag{M.4}$$