The US Debt - Growth Nexus along the Business Cycle

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Abstract

We use the US data gathered by Reinhart and Rogoff (2010) to assess whether debt affects economic growth differently at different phases of the business cycle. In order to do that, we extend the threshold regression model of Chudik et al. (2017) and propose a new threshold quantile ARDL regression model. Our results show that to stimulate growth policy makers can manage the debt/GDP percentage according to how well the economy is doing. The estimated quantile thresholds (range 31-53 per cent) are larger than the one found by Lee et al. (2017) using median regressions, but still (much) smaller than the 90 per cent of Reinhart and Rogoff. In particular, when the US economy observes growth rates above their median value, that is when a smaller debt-to-GDP threshold affects the performance of the economy. In a steady-state situation, in general, regardless of the position of the business cycle and whether the debt-to-GDP ratio is below or above its threshold effect, less debt as a percentage of GDP boosts the US growth. Remarkably, this effect was always greater before than after World War II. Moreover, the most recent decades have witnessed the negative (positive) effect of more (less) debt when the economy had growth rates at their first quartile (median and third quartile). That is, the US policy makers are advised to reduce the debt-to-GDP ratio during expansions to promote growth.

Keywords: Government Debt, Growth, Business Cycle, Threshold Quantile Regression
JEL classification: E6, F34, H60, C22, C24

1 Introduction

Advanced economies such as that of the United States have experienced a long-term increase in public debt as a fraction of GDP. As a result, it has become more difficult to pursue an expansionary policy and recoveries are slower when a financial or economic crisis occurs. To promote potential growth and respond more aggressively to crises we find evidence that policy makers should reduce the debt-to-GDP ratio during expansions.

There are several theories that link economic growth to government debt. Most of the empirical literature shows that debt levels above a particular threshold value penalize growth compared to moderate levels of debt. Reinhart and Rogoff (2010) found a debt threshold of about 90 per cent of GDP and since then the debate has become centred around this value by several other authors using alternative datasets or mean threshold regression models (for example, Caner, Grennes, and Koehler-Geib, 2010, Checherita-Westphal and Rother, 2012, Baum, Checherita-Westphal, and Rother, 2013, Woo and Kumar, 2015, Egert, 2015, Chudik et al., 2017, and Amann and Middleditch, 2020). To put it simple, the threshold literature assumes that the regression model introduces either a jump or a kink at the threshold point. Hansen (2017) estimated a regression kink of around 40 per cent, although Hidalgo, Lee, and Seo (2017) suggest that imposing the kink restriction in the slope is not warranted.[1]

Reinhart and Rogoff's study was definitely a crucial one in shaping the direction that the debate over the debt - growth nexus took for quite some time. Yet the debate is incomplete because, by focusing on the conditional mean growth, it does not explain how the debt level affects (differently) high and low economic growth rates. For instance, when an economy is facing a recession, and thus growing below its average rate, the debt threshold is not necessarily the same as the one that follows from average growth rates. In fact, (un)sustainable debt can hardly be maintained at the same levels whether an economy is in a recession or an expansion. To understand the full picture of the debt - growth nexus, one needs to go beyond the mean growth and study its conditional distribution, namely, the tails.

In this paper we study further the US debt-growth nexus using a threshold-like regression model but going beyond simply assessing the aforementioned so-called Reinhart-Rogoff hypothesis. We use a threshold quantile regression model to see whether high levels of debt affect growth differently at different phases of the business cycle, the whole growth distribution. We compare the periods prior to and after the Second World War event and to overcome the problem of en-

[1] See Panizza and Presbitero (2013) and Eberhardt and Presberito (2015) for additional references to the literature.

dogeneity we follow Chudik et al. (2017)[2] and propose a dynamic (ARDL) model specification (TQARDL). Our main findings are: reducing public debt as an instrument to promote growth has lost some of its effectiveness following World War II; in recessions, reducing debt has no impact on growth, whereas increasing it makes the economy worsen, especially if the debt-to-GDP ratio is below its threshold effect and; in expansions, only reducing debt can impact (positively) the economy.

Only a few papers in the literature use quantile regressions to study the debt - growth nexus and none of them relate it with the business cycle of the economy. Lin (2014) considers a high-dimensional quantile regression model and for that end proposes a threshold quantile Lasso estimator. By including such a large number of determinants, he ends up having a short number of observations (around 45-50), which makes the tail estimation less accurate. For the left tail, the median, and the right tail, Lin (2014) estimates the same threshold level of 38.5%. In our paper we have more than 200 annual observations and find estimated thresholds similar to Lin's but different across quantiles. More recently, covering the period 1946-2009 Lee et al. (2017) estimate the relationship between public debt and median real GDP growth but do not consider the nexus at the tails. Our estimated threshold is larger than the one found by them (of around 30 per cent) but smaller than Reinhart and Rogoff's (about 90 per cent).

We also make a modest methodological contribution to the literature. We extend the TQAR model with a threshold that depends on the quantile proposed by Galvão et al. (2011) to the ARDL specification. Since lagged growth is pre-determined (exogenous), one can apply the standard threshold QR techniques to our model. Nevertheless, it is worth mentioning that in our threshold regression model it is assumed that debt is exogenous and that this follows from the underlying mean regression specification of Chudik et al. (2017). Galvão et al. (2014) and Kuan, Michalopoulos, and Xiao (2017) propose tests in threshold regression models with time series data but for a fixed threshold.

This paper is structured as follows: In Section 2 we present the baseline model as in Chudik et al. (2017) and the more general model that includes a dummy for the event of World War II, and more lagged and interaction terms between the threshold variable and the growth and debt covariates. Section 3 discusses the features of the new threshold quantile regression model (TQARDL), including the estimation and inference methods. Section 4 describes the US data gathered by Reinhart and Rogoff (2010) covering the period 1791-2009 and draws the conclusions about the real GDP growth - debt-to-GDP ratio nexus at different stages of the business cycle.

[2] Alternative methods that deal with the endogeneity of threshold variables and regressors in mean threshold regression models includes Kourtellos, Stengos, and Tan (2016), Seo and Shin (2016), Kourtellos, Stengos, and Sun (2017), and Yu and Phillips (2018), among others.

The findings are corroborated using quarterly data from 1966Q1 to 2019Q4 for total federal debt and federal debt held by the public as percentages of GDP. Section 5 concludes.

2 The General Model

Our baseline model is exactly the same as in Chudik et al. (2017). For a particular country, they specify

$$g_t = \alpha_g + \varphi I\left(d_t > \ln\left(\tau\right)\right) + \delta g_{t-1} + \eta \triangle d_{t-1} + e_t, t = 2, ..., T, \tag{1}$$

where g_t denotes the first difference of the logarithm of real GDP during year t, d_t is the (natural) logarithm of debt-to-GDP ratio, and τ is the threshold level, together with the equation for d_t

$$\triangle d_t = \alpha_d + \rho d_{t-1} + \varkappa \triangle d_{t-1} + \psi g_{t-1} + \varepsilon_t, \tag{2}$$

where the correlated e_t and ε_t are assumed to be serially uncorrelated with zero means. To identify the threshold effect in the output growth equation, they assume that no such threshold effect exists in the debt equation (identification condition). To deal with the simultaneity bias, they assume a linear dependence between the two error terms, $e_t = \kappa \varepsilon_t + u_t$, with ε_t and u_t uncorrelated. The corresponding reduced form equation is given by

$$g_t = c + \varphi I\left(d_t > \ln\left(\tau\right)\right) + \lambda g_{t-1} + \beta_0 \triangle d_t + \beta_1 \triangle d_{t-1} + \beta_2 d_{t-1} + u_t, \tag{3}$$

where $c = \alpha_g - \kappa \alpha_d$, $\lambda = \delta - \kappa \psi$, $\beta_0 = \kappa$, $\beta_1 = \eta - \kappa \varkappa$, and $\beta_2 = -\kappa \rho$, such that u_t and the threshold variable $I\left(d_t > \ln\left(\tau\right)\right)$ are uncorrelated. They further extend this model to the baseline threshold ARDL specification

$$g_t = c + \varphi_1 I\left(d_t > \ln\left(\tau\right)\right) + \varphi_2 \left[I\left(d_t > \ln\left(\tau\right)\right) \times \max\left(0, \triangle d_t\right)\right] + \sum_{l=1}^{p} \lambda_l g_{t-l} + \sum_{l=0}^{p} \beta_l \triangle d_{t-l} + u_t, \tag{4}$$

with $p = 1, 2, 3$ and where the interactive threshold variable takes a nonzero value when d_t exceeds the threshold and the growth of debt-to-GDP is positive. See Chudik et al. (2017) for details.

Very importantly, and contrary to Kourtellos, Stengos, and Tan (2013) and Panizza and Presbitero, (2014), these debt - growth threshold regression models exclude endogeneity, which makes it easier to estimate and make inferences in practice. In fact, the reduced form makes the regressor exogenous but also endogeneity of the threshold variable is absent: "Recall that we have already dealt with the endogeneity of the threshold variable, by considering a panel threshold-ARDL model where the threshold effects are identified by an exclusion restriction and the assumption that output growth and debt error terms are linearly related." (Chudik et al., 2017, pp 138/9).

Despite all the nice features of Chudik et al.'s (2017) models (3) and (4), we extend them further, more in accordance to the relevant literature in this topic. First, we allow for a lagged debt-to-GDP ratio effect at the threshold component, as advocated by Lin (2014). The model selection methods will choose between $I(d_t > \ln(\tau))$ and $I(d_{t-1} > \ln(\tau))$. Second, because the dataset we are using includes observations prior to World War II, we add a dummy variable that captures differences across the two regimes, $w_t = 1$ if t corresponds to a year after 1945. The literature using mean regressions has reported differences between the two periods but for quantile regressions no studies exist to date. Third, we extend Chudik et al. (2017) by including interactions between the threshold variable and the growth and debt covariates. We believe that this is new in the literature. It measures differences at the autoregressive (AR) and distributed lag (DL) components when debt is large compared to when it is small.

The general model we propose is

$$g_t = \phi' x_t + \varphi' x_t I(d_{t-j} > \ln(\tau)) + \pi' x_t w_t + u_t, \text{ with } j = 0, 1, \tag{5}$$

where

$$x_t = (1, g_{t-1}, \triangle d_t, \triangle d_{t-1}, d_{t-1}), \text{ extending (3)}, \tag{6}$$

or

$$x_t = (1, \max(0, \triangle d_t), g_{t-1}, ..., g_{t-p}, \triangle d_t, ..., \triangle d_{t-p}), p = 1, 2, 3, \text{ extending (4)}. \tag{7}$$

The quantities ϕ, φ, and π are unknown parameters and w_t is the dummy variable associated with World War II, defined previously. The way output growth depends on the debt-to-GDP ratio is specified through our model and will be estimated according to the approach we describe next.

3 The Methodology

Threshold mean regression models with exogenous regressors proposed by Chan (1993) and Hansen (2000) have been especially common in debt-growth applications (see several references in the Introduction). However, by focusing exclusively on the conditional mean of the growth distribution, this technique may leave practitioners with a rather incomplete analysis. The (threshold) quantile regression models provide a characterization of the entire conditional distribution of growth given a set of debt regressors. It has the potential to uncover differences in the response of growth to changes in the regressors at different points of its conditional distribution.

In this paper we investigate whether growth quantiles depend on threshold debt levels and if these threshold values are different along the distribution. In other words, our approach is the first to be able to produce a complete picture on how the amount of debt may influence growth at different points throughout the entire business cycle. As mentioned in the Introduction, Lee et al. (2017) studies the median regression but not those quantiles related to expansions and recessions of the economic activity, and Lin (2014) takes different quantiles but never relates the findings with a business cycle analysis, probably due to the lack of observations in his dataset.

The general economic model (5) has a threshold ARDL specification. In this paper we propose a new threshold quantile regression model - the TQARDL - which is nothing but an extension of the work developed by Koenker and Xiao (2006) (the QAR model), Galvão, Montes-Rojas, and Park (2013) (the QARDL model), and Galvão, Montes-Rojas, and Olmo (2011) (the TQAR model). Following this literature, with $\{U_t\}$ a sequence of i.i.d. standard uniform random variables, the TARDL process is defined through

$$g_t = \phi\left(U_t\right)' x_t + \varphi\left(U_t\right)' x_t I\left(d_{t-j} > \ln\left(\tau\left(U_t\right)\right)\right) + \pi\left(U_t\right)' x_t w_t, \text{ with } j = 0, 1, \tag{8}$$

which, by monotonicity of the right-hand side on U_t, it follows that the θ-th conditional quantile function of g_t is written as

$$Q_{g_t}\left(\theta|\mathcal{F}_t\right) = \phi\left(\theta\right)' x_t + \varphi\left(\theta\right)' x_t I\left(d_{t-j} > \ln\left(\tau\left(\theta\right)\right)\right) + \pi\left(\theta\right)' x_t w_t, \text{ with } j = 0, 1, \tag{9}$$

where x_t is (6) or (7) and $\mathcal{F}_t$ is the σ-field generated by $\{g_s, d_s, s \leq t\}$. The monotonicity condition implies that $Q_{g_t}\left(\theta|\mathcal{F}_t\right)$ is monotone increasing in θ for all $\mathcal{F}_t$. See the three previous references for a discussion of the monotonicity of the estimated conditional quantile function. [3]

In this TQARDL model we need to estimate the unknown parameters' function

$$\Lambda\left(\theta\right) = \left(\tau\left(\theta\right), \phi\left(\theta\right)', \varphi\left(\theta\right)', \pi\left(\theta\right)'\right)' : [0,1] \rightarrow \Re^{\dim(\Theta)}, \tag{10}$$

$\widehat{\Lambda}\left(\theta\right)$, and from that obtain the estimated conditional quantile function $\widehat{Q}_{g_t}\left(\theta|\mathcal{F}_t\right)$. With the purpose of studying the US business cycle and how debt influences growth along its different phases, we take the cases of $\theta \in \{0.25, 0.5, 0.75\}$. For $\theta = 0.25$, we interpret the phases of depression (decrease in growth or negative growth) and recovery (increase in growth), and when $\theta = 0.75$ a prosperity (increase in growth) and recession (decrease in growth). The median $\theta = 0.5$ defines the turning points from economic expansion to contraction and vice-versa. We did not estimate the models taking θ closer to the distribution tails because it demands a large number of observations. The data sparsity at the extreme tails makes standard quantile estimators unstable (Li and Wang, 2017).

[3] In our application section, we will compute the estimated conditional quantile functions and show that monotonicity applies. Very interestingly, we performed the same exercise for the UK using the Reinhart and Rogoff dataset for the period 1831-2009 but concluded that the monotonicity assumption might not, in fact, be holding. This deserves further attention as mentioned in the Conclusion.

The estimation of the TQARDL model with a non-common threshold (9) is a two-stage procedure similar to that in Galvão, Montes-Rojas, and Olmo (2011) in which we now add the DL terms present in Galvão, Montes-Rojas, and Park (2009, 2013). The point estimators $\widehat{\Lambda}_2(\theta) = \left(\widehat{\phi}(\theta)', \widehat{\varphi}(\theta)', \widehat{\pi}(\theta)'\right)'$ and $\widehat{\tau}(\theta)$ are the

$$\arg\min_{(\Lambda_2,\tau)} S_{T\theta}(\Lambda_2,\tau) \left[= \sum_{t=1}^{T} \rho_\theta \left(g_t - X_t(\tau(\theta))' \Lambda_2(\theta)\right)\right], \tag{11}$$

where $X_t(\tau(\theta))' = (x_t', x_t' I(d_{t-j} > \ln(\tau(\theta))), x_t' w_t)$, with $j = 0, 1$, and $\rho_\theta(e) = e(\theta - I(e < 0))$. The criterion $S_{T\theta}(\Lambda_2,\tau)$ is non-convex, so the minimizer cannot be found by the standard quantile algorithms. Instead, we can consider the profile quantile regression and compute through concentration, as in Koenker and Bassett (1978), because $S_{T\theta}(\Lambda_2,\tau)$ is convex in Λ_2.

For fixed θ, first, conditional on $\tau(\theta)$, obtain $\widehat{\Lambda}_2(\tau(\theta)) = \arg\min_{\Lambda_2} S_{T\theta}(\Lambda_2,\tau)$ as in standard QR estimation. Then, estimate $\tau(\theta)$ by grid search over $\Gamma = [\tau_L, \tau_U]$, i.e.,

$$\widehat{\tau}(\theta) = \arg\min_{\tau\in\Gamma} S_{T\theta}(\tau) \left[= S_{T\theta}\left(\widehat{\Lambda}_2(\tau), \tau\right)\right], \tag{12}$$

as in typical threshold regression estimation. Once $\widehat{\tau}(\theta)$ is obtained, compute $\widehat{\Lambda}(\theta) = \left(\widehat{\tau}(\theta), \widehat{\Lambda}_2(\widehat{\tau}(\theta))'\right)'$, which yields the solution to $S_{T\theta}(\Lambda_2,\tau)$. Here, $S_{T\theta}(\tau)$ takes on less than $T-1$ distinct values (from $d_{t-j}, j = 1, t = 2, ..., T$) and therefore the admissible set Γ includes observed $d_{t-j}, j = 0, 1$, after trimming at 15%, say, upper and lower values (τ_L, τ_U).

The proof for consistency and asymptotic distribution of the parameter estimators are derived as in Galvão, Montes-Rojas, and Olmo (2011). In fact, extending the TQAR to TQARDL implies adding assumptions on the DL processes (see Galvão, Montes-Rojas, and Park, 2009, 2013) but preserves the results reported in Galvão, Montes-Rojas, and Olmo (2011), after the appropriate choice of the model's notation. In terms of the data, ones needs weak stationarity, which is something that the standard unit root tests confirm for growth and debt-to-GDP ratio time differences. With respect to the model specification, we assume that debt is strictly exogenous thus trusting that by adopting Chudik et al. (2017) setup we do not have to deal with endogeneity in our TQARDL model. Addressing this issue is out of the scope of this paper so we shall come back to it at the Conclusion.

In a nutshell, pointwise (for fixed θ), $\widehat{\Lambda}_2(\theta)$ and $\widehat{\tau}(\theta)$ are shown to be consistent; for fixed θ and τ, $\widehat{\Lambda}_2(\tau(\theta))$ is $\sqrt{T}-$asymptotically normal; and it remains to study the asymptotic distribution of the threshold parameter estimator $\widehat{\tau}(\theta)$, which is still an unsolved question in the time series literature.[4]

[4] This is a theoretical issue in time series econometrics that is clearly out of the scope of this paper. Important references on this topic include Galvão et al. (2014) and Su and Xu (2017), among others. Galvão et al. (2014)

Despite all these limitations, the tools we have are more than enough to tackle this paper's purpose, which is new in this kind of literature: being able to make inferences on how the US debt determines its economic growth and consistently estimate debt thresholds along the different phases of the business cycle.

4 Empirical Application

The estimation of our TQARDL models with a non-common threshold requires long span time series data of real GDP growth and debt-to-GDP ratio, which are available at most for only one or two countries in the world. Following Hansen (2017), and thereby extending the period of Lee et al. (2017), we use the US data gathered by Reinhart and Rogoff (2010) and posted on their website. The data for the US provide the most complete known set covering the period 1791-2009, so that there are $T = 219$ annual observations and plenty of complete business cycles. These data end in the aftermath of the financial crisis and economic recession of 2008/2009.

We conducted the usual unit root tests (Phillips-Perron, Ng-Perron, among others) and the results suggest that growth, g_t, is stationary just as d_t, the logarithm of debt-to-GDP ratio. Therefore, taking first differences of the logarithm of debt-to-GDP ratio implies over-differentiation of this variable and an estimated model with a poor fit, and in which the debt covariates are not statistically significant. As an alternative, and following Hansen (2017) and Lee et al. (2017), we tested for the debt-to-GDP ratio in levels and concluded that it is non-stationary, so that the first-differences in the model are now stationary[5]. In sum, our empirical application uses growth and debt in percentage points:

$$g_t = \phi' x_t + \varphi' x_t I\left(d_{t-j} > \tau\right) + \pi' x_t w_t + u_t, \text{ with } j = 0, 1, \tag{13}$$

where g_t denotes the first difference of the logarithm of real GDP and d_t the debt-to-GDP ratio. Next, we refer to the original Chudik et al. (2017) models (3) and (4) as CMPR1 and CMPR2 (for a given p), respectively, and to our extended models (6) and (7), for $j = 0$ or $j = 1$, as CMPR1ext and CMPR2ext (for a given p), respectively. As mentioned above, we choose between $j = 0$ or $j = 1$ by means of the usual model selection methods. In terms of selecting p, we also implement the standard tests for statistical significance.[6]

propose a test statistic for linearity, but it has a limiting distribution that depends on unknown parameters, as well as not covering the modelling and estimation issues in TQAR models with a non-common threshold. On the other hand, Su and Xu's (2017) setup considers only i.i.d. observations

[5] In fact, the models of Hansen (2017) and Lee et al. (2017) have the debt-to-ratio in levels, thus with a mixture of stationary (growth) and nonstationary (debt) time series in it. In that respect, we follow Chudik et al. (2017) by including debt in first-differences.

[6] In their paper, Chudik et al. (2017) present the results for the different $p = 0, 1, 2, 3$.

4.1 Estimated Threshold Effects

With a very few exceptions, the literature cited herein has modelled the debt-growth nexus through mean regressions. For this reason, and for comparison matters, we estimate threshold mean ARDL models before the proposed threshold quantile ARDL models. The econometric approach follows closely the work by Hansen (1996, 1999, and 2000). The estimated debt-to-GDP ratio threshold τ ranges from 33% (CMPR2ext model with $p = 2$ and $j = 1$) to 56% (CMPR2 model with $p = 2$), which is (much) smaller than those found in the standard literature using mean regressions, and for debt levels above the threshold the estimated mean growth gain $\varphi_{intercept}$ ranges from 2.1% (CMPR2ext with $p = 2$ and $j = 1$) to 3.7% (CMPR1).[7]

The scope of this paper includes studying the debt-growth nexus along the whole business cycle, something that the mean regression does not capture. Hence, following the econometric approach described in Section 3, we now present the estimated TQARDL models for quantiles $\theta = 0.25$, $\theta = 0.50$, and $\theta = 0.75$. Our quantile regression models include the AR dynamic terms. Therefore, to prevent endogeneity in the model we test for autocorrelation as in Huo et al (2017). We confirm the null hypothesis of having no autocorrelated errors at a 5% level (the CMPR2 model for $\theta = 0.25$ and the CMPR1ext for all quantiles, only at a 1% level). The estimated threshold effects are in Table 1.

Exactly as in the mean regression models, the smallest TQARDL estimated threshold τ is found for the CMPR2ext model and the largest for the CMPR2 model. Our estimated threshold is larger than the one found by Lee et al. (2017) using median regressions, with the smallest being $\widehat{\tau} = 31\%$ for $\theta = 0.25$, $\widehat{\tau} = 33\%$ for $\theta = 0.50$, and $\widehat{\tau} = 32\%$ for $\theta = 0.75$ for the CMPR2ext model. For the other extended model (CMPR1ext), the estimated debt-to-GDP ratio threshold decreases with the conditional quantile growth rates: $\widehat{\tau} = 43\%$ for $\theta = 0.25$, $\widehat{\tau} = 40\%$ for $\theta = 0.50$, and $\widehat{\tau} = 33\%$ for $\theta = 0.75$. This finding in which the estimated threshold parameter falls with the quantile is corroborated by the CMPR2 model. This means that it is when the US economy is doing well, with growth rates above their median value, that a smaller debt-to-GDP threshold affects the performance of the economy.

With respect to the threshold effect intercept, $\varphi_{intercept}$, we find positive point estimates for model CMPR2ext, negative ones for CMPR2, and both signs for CMPR1 and CMPR1ext (positive when $\theta = 0.25$ and $\theta = 0.50$ and negative for $\theta = 0.75$). In all cases we obtain (in

[7] For CMPR1 we found $\widehat{\tau} = 38$ and for CMPR1ext (with $j = 0$) $\widehat{\tau} = 40$. Moreover, $\widehat{\varphi}_{intercept} = 2.4$ for CMPR1ext (with $j = 0$). On the contrary, CMPR2 (with $p = 2$) estimates a negative $\varphi_{intercept}$ (-2.7). For the largest model, CMPR2ext, besides $\varphi_{intercept}$, we find statistical significance for the threshold effects $\varphi_{g_{t-1}}, \varphi_{g_{t-2}}, \varphi_{\triangle d_t}$ and $\varphi_{\triangle d_{t-1}}$.

Table 1: Estimated Threshold Effects - TQARDL Models

		θ		
		0.25	0.50	0.75
CMPR1	τ	35.932	35.932	46.973
	$\varphi_{intercept}$	4.482^{***} (1.593)	2.045^{*} (1.134)	-2.191^{*} (1.244)
CMPR1ext	τ	43.164	39.962	32.973
	$\varphi_{intercept}$	10.887^{***} (3.416)	9.030^{***} (2.838)	-8.251^{**} (3.911)
	$\varphi_{g_{t-1}}$	0.606^{**} (0.261)		
	$\varphi_{\triangle d_t}$	1.156^{***} (0.267)	0.826^{***} (0.364)	0.944^{***} (0.311)
	$\varphi_{\triangle d_{t-1}}$			-0.924^{***} (0.167)
	$\varphi_{d_{t-1}}$	-0.175^{***} (0.044)	-0.114^{***} (0.045)	0.367^{***} (0.105)
CMPR2	τ	53.041	52.122	51.022
	$\varphi_{intercept}$	-1.410 (1.123)	-1.240^{*} (0.673)	-2.217^{***} (0.687)
	$\varphi_{\max(0,\triangle d_t)}$	0.389 (0.291)	0.340 (0.317)	0.481^{**} (0.219)
CMPR2ext	τ	30.848	33.143	32.111
	$\varphi_{intercept}$	1.676^{***} (0.250)		5.595^{***} (2.005)
	$\varphi_{\max(0,\triangle d_t)}$			-1.405^{**} (0.645)
	$\varphi_{g_{t-1}}$	0.490^{***} (0.103)	0.473^{***} (0.137)	0.558^{***} (0.171)
	$\varphi_{g_{t-2}}$	-0.404^{***} (0.151)		
	$\varphi_{\triangle d_t}$		0.537^{**} (0.215)	0.887^{**} (0.432)
	$\varphi_{\triangle d_{t-1}}$		-0.321^{**} (0.161)	
	$\varphi_{\triangle d_{t-2}}$	-0.392^{***} (0.090)		

Notes: CMPR1ext with $j = 0$, CMPR2 with $p = 2$, CMPR2ext with $p = 2, j = 1$

*,**,*** stands for statistically significant at 10%, 5%, 1% levels, respectively.

absolute value) larger estimates than those reported in Lee et al. (2017) of negative 1 percentage point. We shall consider our more general extended models CMPR1ext and CMPR2ext. For the former model, estimated growth increases at the threshold debt level for $\theta = 0.25$ (10.9%) and $\theta = 0.50$ (9%) and decreases by 8.3% at the largest quantile $\theta = 0.75$. For the latter model, the estimated gain in growth is 1.7% for $\theta = 0.25$ and 5.6% for $\theta = 0.75$. Thus, we have contradictory results for $\theta = 0.75$, but at least for $\theta = 0.25$ one can conclude that when the US economy is struggling, larger debt-to-GDP ratios (of at least about 43%) tend to promote growth, whose gains can be greater than 1.7% per year. To the best of our knowledge this a new and important finding for this literature.

4.2 The Aftermath of the 2009 Crisis

In this section we examine the relevance of our models by studying the aftermath of the 2009 financial crisis. We estimated the TQARDL models using data up to year 2009 and now we evaluate their ability to explain the effects of debt on growth in 2010. Of the four models presented earlier, we focus on the extended model CMPR2ext. We use the extended version of the original Chudik et al. (2017) models because we found statistical significance of w_t (world war) and take CMPR2ext rather than CMPR1ext because it is the most general model. In fact, the pseudo-R2 goodness of fit measure suggested by Koenker and Machado (1999) is the largest for model CMPR2ext at any of the quantiles of interest.

For the PostWW period, the estimated TQARDL CMPR2ext models are:

$$\widehat{Q}_{g_t}(0.25|\mathcal{F}_t) = \begin{cases} 1.697 - 2.262\max(0, \triangle d_t) - 0.539\triangle d_t + 0.690\triangle d_{t-1}, \\ \text{if } d_{t-1} \leq 30.848 \\ 1.697 - 0.586\max(0, \triangle d_t) + 0.490 g_{t-1} - 0.404 g_{t-2} - \\ 0.539\triangle d_t + 0.690\triangle d_{t-1} - 0.392\triangle d_{t-2}, \text{ if } d_{t-1} > 30.848 \end{cases}, \quad (14)$$

$$\widehat{Q}_{g_t}(0.50|\mathcal{F}_t) = \begin{cases} 2.213 + 0.192\max(0, \triangle d_t) - 0.183 g_{t-2} - 1.106\triangle d_t + \\ 0.659\triangle d_{t-1} - 0.199\triangle d_{t-2}, \text{ if } d_{t-1} \leq 33.143 \\ 2.213 + 0.192\max(0, \triangle d_t) + 0.473 g_{t-1} - 0.183 g_{t-2} - \\ 0.569\triangle d_t + 0.338\triangle d_{t-1} - 0.199\triangle d_{t-2}, \text{ if } d_{t-1} > 33.143 \end{cases}, \quad (15)$$

$$\widehat{Q}_{g_t}(0.75|\mathcal{F}_t) = \begin{cases} -2.641 + 2.051\max(0, \triangle d_t) - 0.274 g_{t-2} - 1.619\triangle d_t + \\ 0.494\triangle d_{t-1} - 0.425\triangle d_{t-2}, \text{ if } d_{t-1} \leq 32.111 \\ 2.954 + 0.646\max(0, \triangle d_t) + 0.558 g_{t-1} - 0.274 g_{t-2} - \\ 0.732\triangle d_t + 0.494\triangle d_{t-1} - 0.425\triangle d_{t-2}, \text{ if } d_{t-1} > 32.111 \end{cases}. \quad (16)$$

By approximating the dynamic AR(2) components, $\frac{1}{1-aL-bL^2}$, to a lag polynomial of order 8[8] (the maximum duration of a complete business cycle measured in years), $1+cL+dL^2+...+eL^8$, and taking in the model the observed debt levels until 2009, we obtain

$$\widehat{Q}_{g_{2010}}(0.25|\triangle d_{2010}) = 4.4612 - 0.586 \max(0, \triangle d_{2010}) - 0.539 \triangle d_{2010}, \tag{17}$$

$$\widehat{Q}_{g2010}(0.50|\triangle d_{2010}) = 8.8048 + 0.192 \max(0, \triangle d_{2010}) - 0.569 \triangle d_{2010}, \tag{18}$$

$$\widehat{Q}_{g_{2010}}(0.75|\triangle d_{2010}) = 10.732 + 0.646 \max(0, \triangle d_{2010}) - 0.732 \triangle d_{2010}. \tag{19}$$

This prooves, at least for this case of interest, the presence of monotonicity at the estimated g_{2010} quantile function for any value of $\triangle d_{2010}$ that is economically relevant, $\widehat{Q}_{g_{2010}}(0.25|\triangle d_{2010}) < \widehat{Q}_{g_{2010}}(0.50|\triangle d_{2010}) < \widehat{Q}_{g_{2010}}(0.75|\triangle d_{2010})$.

According to the estimated models for growth in 2010, had the US economy reduced its debt, it would have increased growth at any quantile. Or at least if the debt level had not changed, growth could have ranged from 4.5% (first quartile) to 10.7% (third quartile). But because of the financial crisis, after an increase of 10.1% in 2008 and 9.4% in 2009, the US economy saw another increase, now of 8% in 2010, $\triangle d_{2010} = 8$. For this case, the models predicted $\widehat{Q}_{g_{2010}}(0.25|\triangle d_{2010} = 8) = -4.5\%$, $\widehat{Q}_{g_{2010}}(0.50|\triangle d_{2010} = 8) = 5.8\%$, and $\widehat{Q}_{g_{2010}}(0.75|\triangle d_{2010} = 8) = 10\%$, which compares to the growth actually observed in 2010 of 3.9% in nominal terms (close to but still bellow $\widehat{Q}_{g_{2010}}(0.50|\triangle d_{2010} = 8)$). As a conclusion, our model was in some way able to anticipate the recovery of the US economy in the aftermath of the 2009 financial crisis.

4.3 The Debt-Growth Nexus Along the Business Cycle

In the previous sections, we focused on particular aspects of the estimated models such as the threshold effects and the ability to explain the aftermath of the 2009 crisis. We now make use of the full information the estimated TQARDL CMPR2ext models provide with the purpose of better understanding how different phases of the US business cycle are affected by the debt-to-GDP threshold and variations. The model of interest (5) where x_t is defined as (7) says that growth depends on its past, debt-to-GDP ratio (levels and variations, current and past), and distinguishes the pre- and post-WWII periods.

We estimated quantile models and therefore growth is affected by these variables but differently depending on whether it is observing "small", "median", or "large" growth rates ($\theta = 0.25, 0.5, 0.75$, respectively). Very noticeably, for any particular position of the business cycle, θ, changes at the debt-to-GDP ratio may push growth upward or downward. In this paper

[8] The coefficients associated with lags greater than 8 are essentially zero, meaning that the approximation we use is quite good.

we characterize the debt-growth nexus along the business cycle in its steady-state path. That is, we do not study the estimated TQARDL models in their short-term dynamics but instead look at the estimated relationships in their equilibrium path. In particular, we evaluate growth g_t at the equilibrium level g^* and do the same for the other variables of the models, $d^*, \triangle d^*, x^*$. Dropping "$*$" for the sake of simplicity of notation, we obtain for the pre-WWII period

$$\widehat{Q}_g\left(0.25|d,\triangle d\right)=\begin{cases} 1.697-1.314\triangle d, \text{ if } d\leq 30.848 \\ 1.697+0.086g-0.030\triangle d, \text{ if } d>30.848, \triangle d>0 \\ 1.697+0.086g-1.706\triangle d, \text{ if } d>30.848, \triangle d\leq 0 \end{cases}, \tag{20}$$

$$\widehat{Q}_g\left(0.50|d,\triangle d\right)=\begin{cases} 4.130-0.183g-0.410\triangle d, \text{ if } d\leq 33.143, \triangle d>0 \\ 4.130-0.183g-1.305\triangle d, \text{ if } d\leq 33.143, \triangle d\leq 0 \\ 4.130+0.290g-0.194\triangle d, \text{ if } d>33.143, \triangle d>0 \\ 4.130+0.290g-1.089\triangle d, \text{ if } d>33.143, \triangle d\leq 0 \end{cases}, \tag{21}$$

$$\widehat{Q}_g\left(0.75|d,\triangle d\right)=\begin{cases} 5.612-0.274g+0.001\triangle d, \text{ if } d\leq 32.111, \triangle d>0 \\ 5.612-0.274g-2.044\triangle d, \text{ if } d\leq 32.111, \triangle d\leq 0 \\ 11.207+0.284g-0.511\triangle d, \text{ if } d>32.111, \triangle d>0 \\ 11.207+0.284g-1.157\triangle d, \text{ if } d>32.111, \triangle d\leq 0 \end{cases}, \tag{22}$$

and for the post-WWII period

$$\widehat{Q}_g\left(0.25|d,\triangle d\right)=\begin{cases} 1.697-2.111\triangle d, \text{ if } d\leq 30.848, \triangle d>0 \\ 1.697+0.151\triangle d, \text{ if } d\leq 30.848, \triangle d\leq 0 \\ 1.697+0.086g-0.827\triangle d, \text{ if } d>30.848, \triangle d>0 \\ 1.697+0.086g-0.241\triangle d, \text{ if } d>30.848, \triangle d\leq 0 \end{cases}, \tag{23}$$

$$\widehat{Q}_g\left(0.50|d,\triangle d\right)=\begin{cases} 2.213-0.183g-0.454\triangle d, \text{ if } d\leq 33.143, \triangle d>0 \\ 2.213-0.183g-0.646\triangle d, \text{ if } d\leq 33.143, \triangle d\leq 0 \\ 2.213+0.290g-0.238\triangle d, \text{ if } d>33.143, \triangle d>0 \\ 2.213+0.290g-0.430\triangle d, \text{ if } d>33.143, \triangle d\leq 0 \end{cases}, \tag{24}$$

$$\widehat{Q}_g\left(0.75|d,\triangle d\right)=\begin{cases} -2.641-0.274g+0.501\triangle d, \text{ if } d\leq 32.111, \triangle d>0 \\ -2.641-0.274g-1.550\triangle d, \text{ if } d\leq 32.111, \triangle d\leq 0 \\ 2.954+0.284g-0.017\triangle d, \text{ if } d>32.111, \triangle d>0 \\ 2.954+0.284g-0.663\triangle d, \text{ if } d>32.111, \triangle d\leq 0 \end{cases}. \tag{25}$$

As we see, at each θ growth g responds differently according to whether d is bellow or above the threshold value $\widehat{\tau}$ after a change of d, negative or positive. In Table 2 we provide for each of the three positions of the US business cycle the estimated change of growth, $\triangle g$, as a response of a

Table 2: Change of Growth Given a Unit Variation of the Debt-to-GDP Ratio

		$\triangle g$			
	θ	$d \leq \widehat{\tau}, \triangle d = -1$	$d \leq \widehat{\tau}, \triangle d = 1$	$d > \widehat{\tau}, \triangle d = -1$	$d > \widehat{\tau}, \triangle d = 1$
pre-WW	0.25	1.314	-1.314	1.866	-0.032#
	0.50	1.103	-0.346	1.533	-0.273#
	0.75	1.604	0.001#	1.615	-0.713
post-WW	0.25	-0.151#	-2.111	0.263#	-0.904
	0.50	0.546	-0.383#	0.605	-0.335#
	0.75	1.217	0.393#	0.926	-0.024#

Notes: # stands for not statistically significant

unit change of the debt-to-GDP ratio, $|\triangle d| = 1$. To illustrate these impacts along the business cycle we present Figures 1 and 2 for the pre-WWII period and 3 and 4 for the post-WWII period. Figures 1 and 3 show how the empirical CDF of growth may shift as a result of changes of the debt-to-GDP ratio and Figures 2 and 4 position the effects of the debt-to-GDP ratio policy along the US business cycle.

Several general conclusions can be drawn from these results. Firstly, with the exception of small growth rates for the post-WWII period in which the effect is not statistically significant, regardless of the position of the business cycle and whether the debt-to-GDP ratio is below or above its threshold effect, less debt as a percentage of GDP stimulates growth. In the other way around, greater debt ratio never benefits the US growth. In fact, in the post-WWII period it can impair growth when the economy is struggling ($\theta = 0.25$), and before the war if the debt-to-GDP ratio was below its threshold effect, it penalized growth at its small or median values and, if the ratio was above, it penalized growth when the economy was doing well ($\theta = 0.75$).

Secondly, the (non-negative) effect on growth following a fall in the debt-to-GDP ratio was always greater before than after the war. The (non-positive) effect on growth occurred in general after WWII. Moreover, when the economy was performing well, $\theta = 0.75$, and during the turning point from economic expansion to contraction (or vice-versa), $\theta = 0.5$, the impact on growth was greater before the war. That is, the US economy after WWII found ways to minimize the impact of debt on growth.

Thirdy, we observe more positive statistically significant effects after a fall of the debt-to-GDP ratio than negative ones. Before the war we see no pattern of change in growth as we move along the business cycle, $\theta \in \{0.25, 0.5, 0.75\}$. But after the war we notice that as we go from the phase of depression, $\theta = 0.25$, to recovery, $\theta = 0.5$, and finally to prosperity, $\theta = 0.75$, the

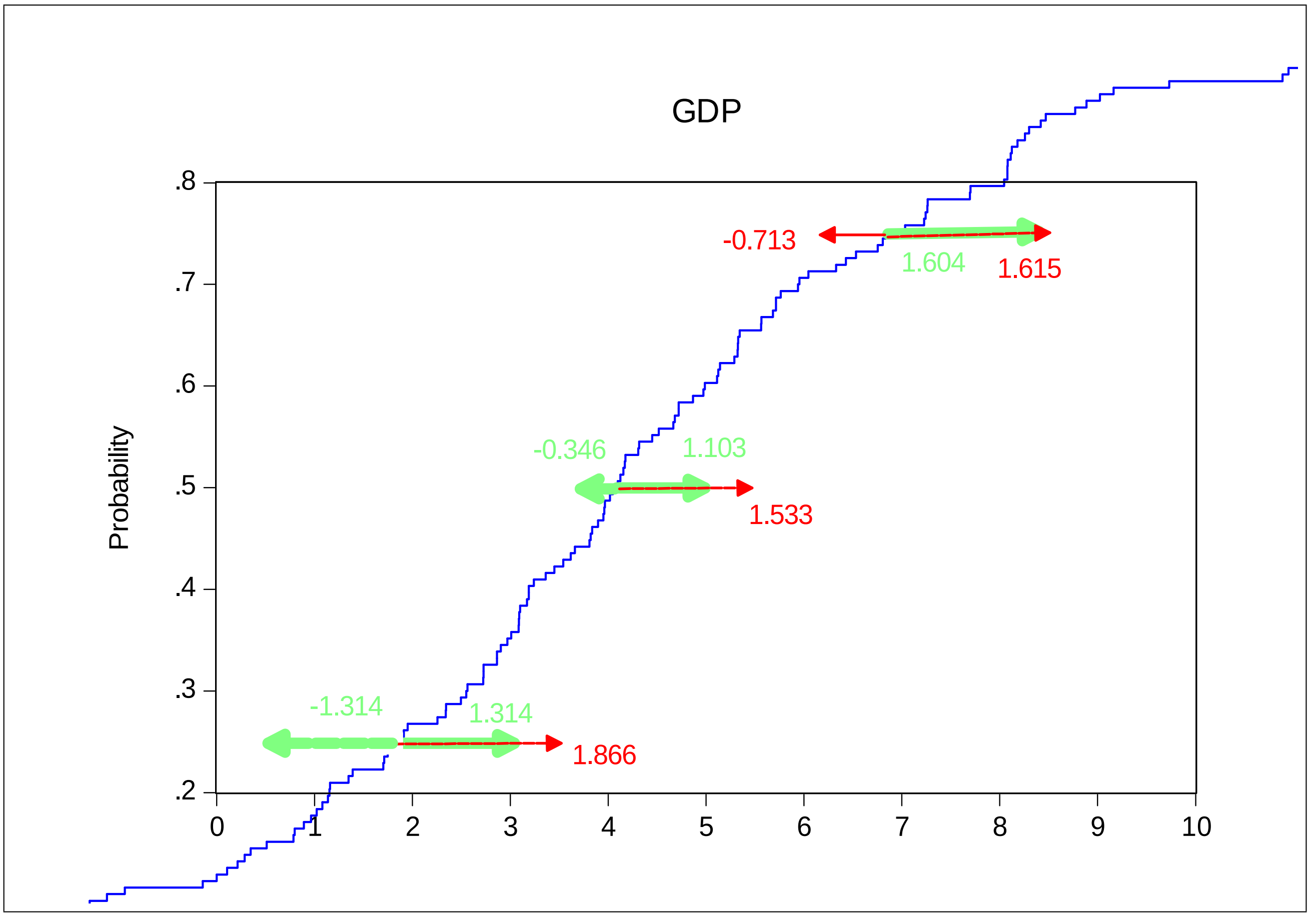

Figure 1: Empirical CDF of growth and impact on growth at each quantile for the pre-WWII period (thin: debt-to-GDP ratio above the threshold; thick: debt-to-GDP ratio below the threshold)

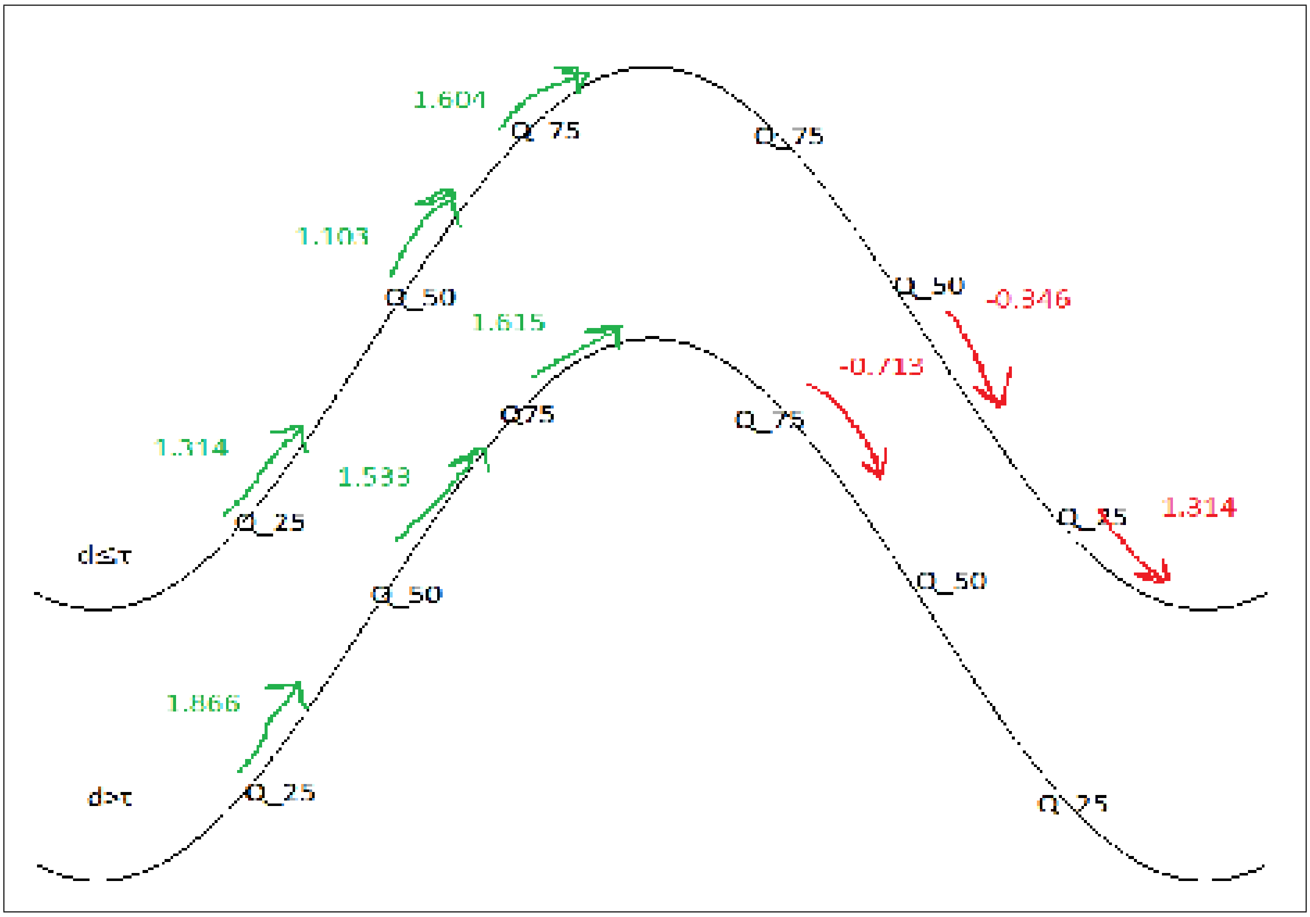

Figure 2: Impact on growth along the US Business Cycle for the pre-WWII period (arrows up: unit decrease of the debt-to-GDP ratio; arrows down: unit increase of the debt-to-GDP ratio)

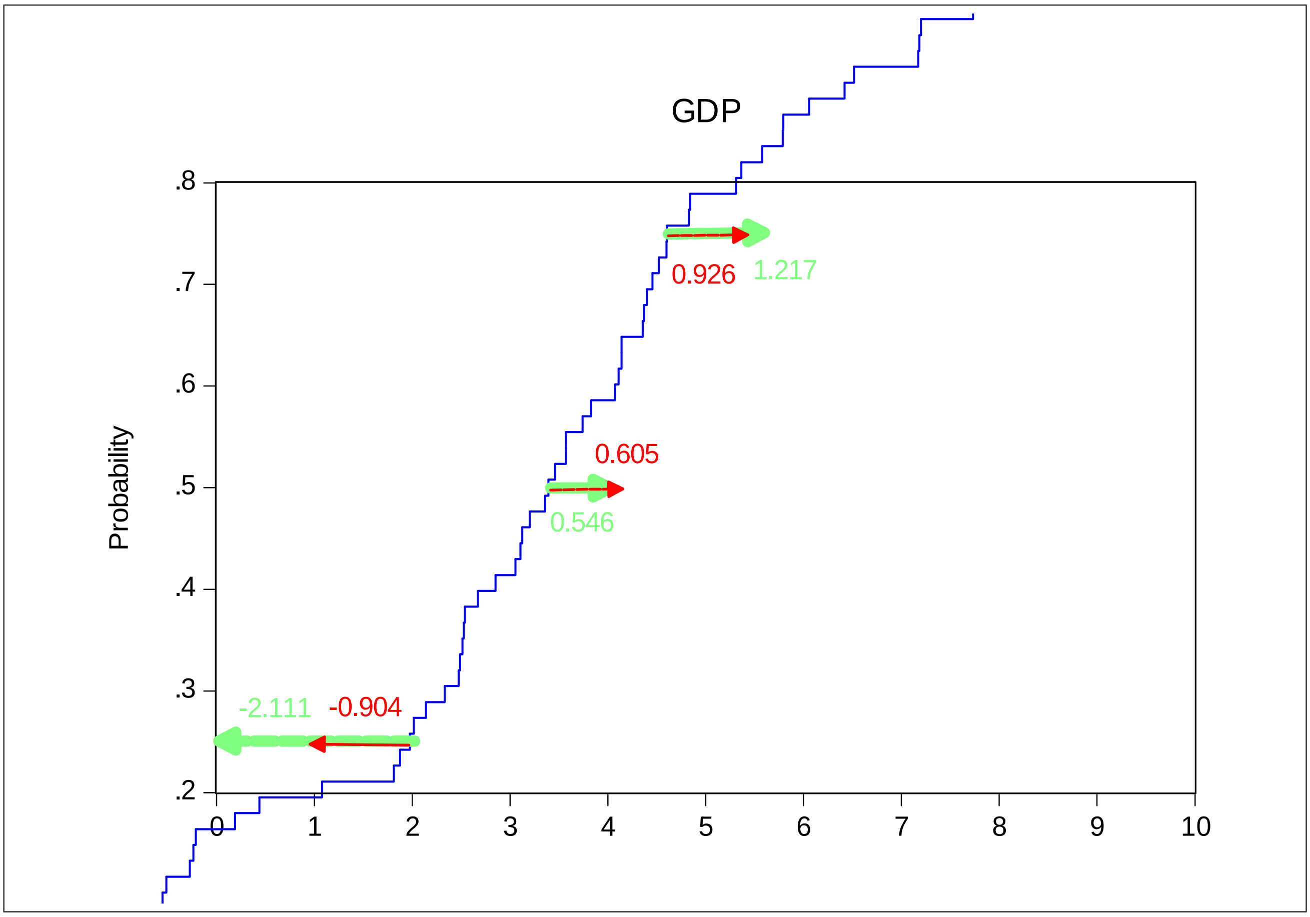

Figure 3: Empirical CDF of growth and impact on growth at each quantile for the post-WWII period (thin: debt-to-GDP ratio above the threshold; thick: debt-to-GDP ratio below the threshold)

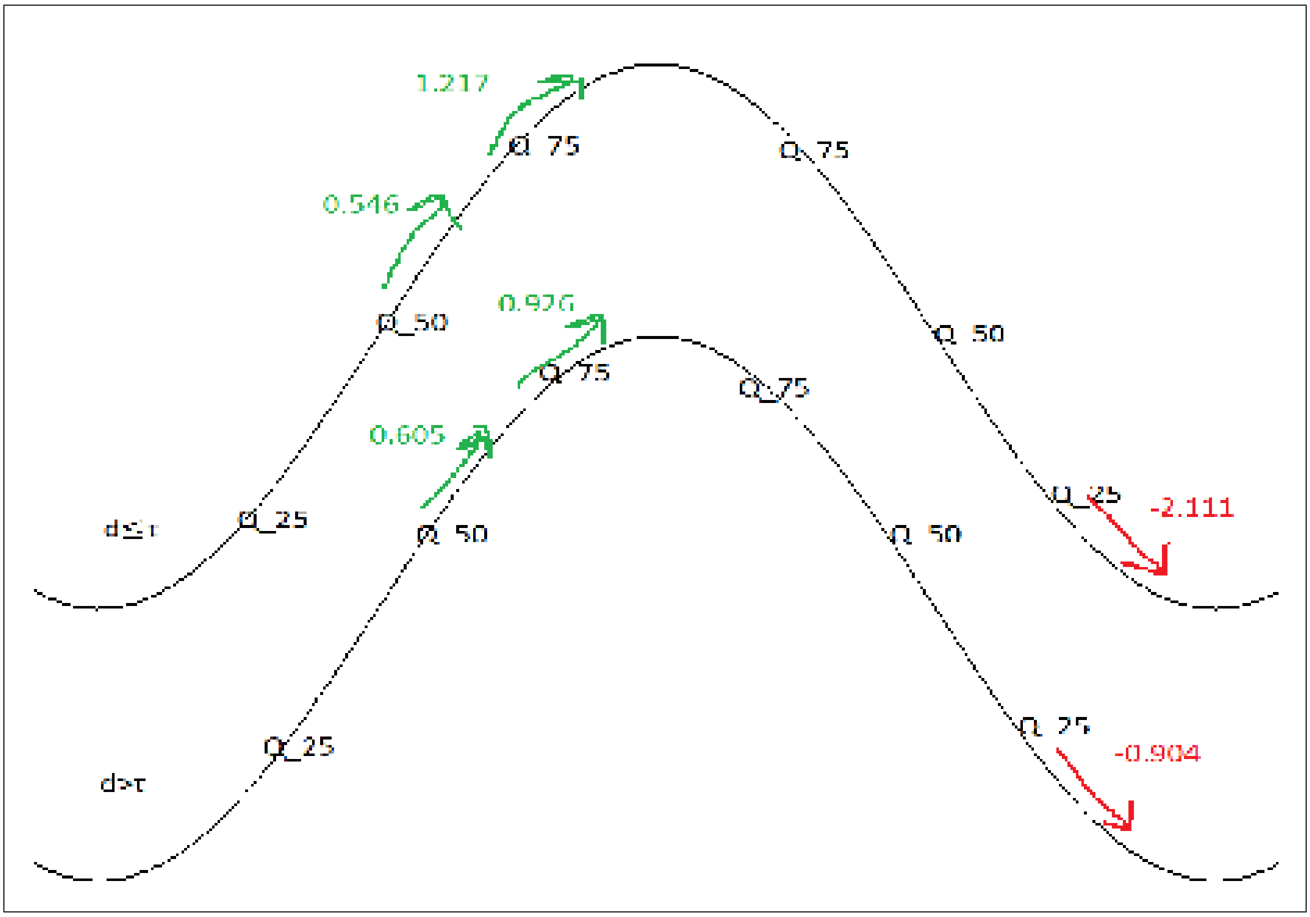

Figure 4: Impact on growth along the US Business Cycle for the post-WWII period (arrows up: unit decrease of the debt-to-GDP ratio; arrows down: unit increase of the debt-to-GDP ratio)

change in growth becomes greater (the negative cases even become statistically not significant). Apparently, the most recent decades witnessed the negative (positive) effect of more (less) debt when the economy had small (median and high) growth rates.

4.4 The Debt-Growth Nexus Using Quarterly Data

The yearly data gathered by Reinhart and Rogoff (2010) covers more than two centuries of data which is adequate for the techniques employed and for studying the long-term economic business cycles. In this subsection, as a robustness check exercise, we use intra annual data obtained from the FRED (St. Louis Fed) website https://fred.stlouisfed.org/categories/5. The quarterly data includes a smaller sample period starting in 1966Q1 and ending 2019Q4, prior to the Covid-19 pandemic situation (total of $T = 216$ observations). Notice that 1966Q1-2019Q4 does not coincide to the previously studied post-WWII period (1946-2009).

As before, we consider the most general model CMPR2ext with $p = 2$:

$$g_t = \phi' x_t + \varphi' x_t I\left(d_{t-j} > \tau\right) + u_t, \text{ with } j = 0, 1, \tag{26}$$

where

$$x_t = \left(1, \max\left(0, \triangle d_t\right), g_{t-1}, g_{t-2}, \triangle d_t, \triangle d_{t-1}, \triangle d_{t-2}\right) \tag{27}$$

and measure g and d as in the St. Louis Fed's website - real GDP growth (percent change from quarter one year ago) and total federal debt as percent of GDP, respectively. For matter of completeness we also consider three different subtypes of debt-to-GDP ratios which start in 1970Q1 ($T = 200$): federal debt held by the public, federal debt held by private investors, and federal debt held by foreign and international investors. [9]Figure 5 shows the growth and debt time series.

For the mean threshold regression model, the estimated total debt-to-GDP ratio threshold τ is basically the same as the one with yearly data (35% for quarterly against 33% for yearly). But contrary to yearly data, the best fit arises from a threshold variable with $j = 0$. The estimated thresholds equal 31% for debt held by the public and 21% for debt held by private investors. There is no evidence of a threshold model for debt held by foreign and international investors (instead of a TARDL model it is a ARDL-type model).

[9] There is also data for federal debt held by agencies and trusts and held by federal reserve banks. For the agencies and trusts there is no evidence of a threshold regression. We do not consider the debt held by federal reserve banks because there is very little variation until about 2009 (around 5%) and then because of quantitative easing it rose steadily until mid 2010's, thus against the idea of a model specification that includes a threshold term.

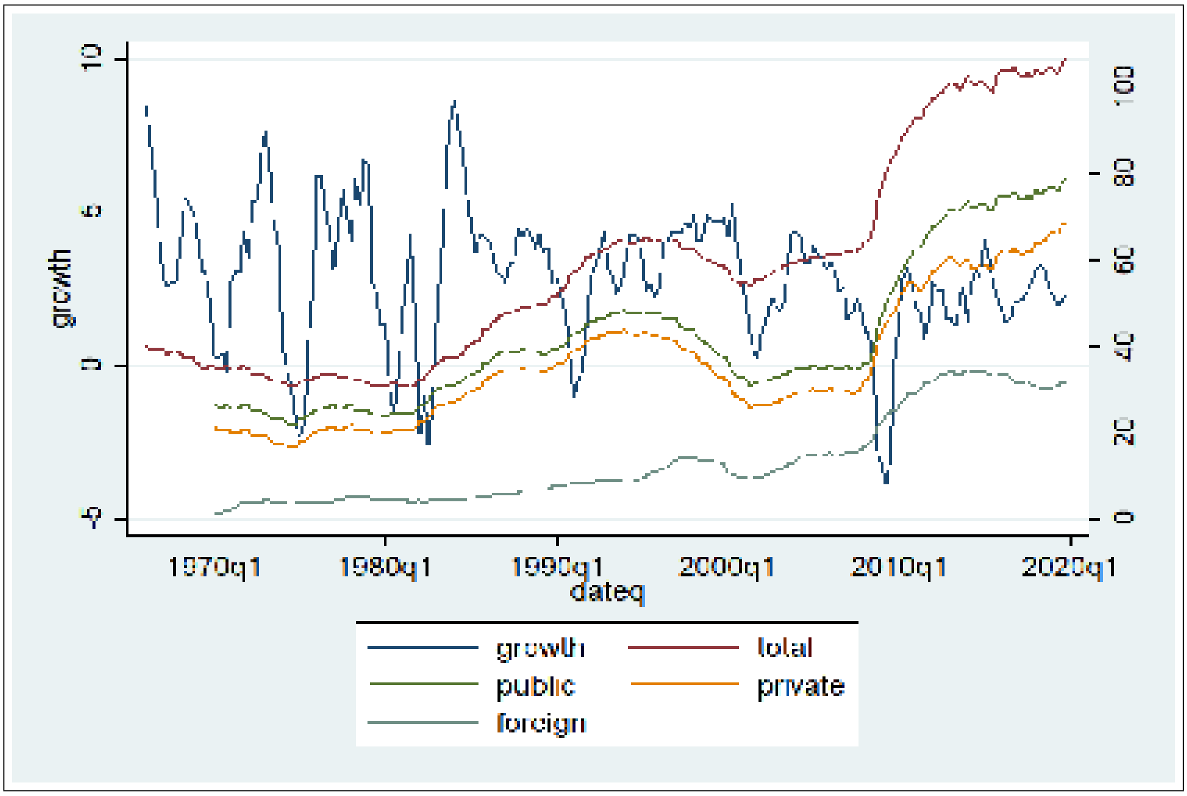

Figure 5: Growth and Debt Time Series

Next, we present in Table 3 the estimated TQARDL models for quantiles $\theta = 0.25$, $\theta = 0.50$, and $\theta = 0.75$. We do not show results for debt held by private investors because the observed time series is about the same as debt held by the public (the difference is at the value of the estimated threshold - see above) nor for debt held by foreign and international investors because there was no evidence of a threshold term in the TARDL model.

The estimated thresholds $\widehat{\tau}$ across quantiles $\theta = 0.25, 0.50, 0.75$ using total debt are very similar to those obtained with yearly data. With respect to debt held by the public we conclude that $\widehat{\tau}$ at the median is the same as for the mean regression (31%) but it is larger for $\theta = 0.75$ (35%) and smaller for $\theta = 0.25$ (26%). That is, the growth-debt ADL model changes at a larger level of debt held by the public during expansions compared to recessions. An important difference between yearly and quarterly data is the fact that with the intra annual data the threshold effect intercept, $\varphi_{intercept}$, is not statistically significant at any θ. The model's intercept is the same for debt levels below or above its correspondent threshold: $\widehat{\phi}_{intercept}\,(\theta = 0.25)$ =0.460, $\widehat{\phi}_{intercept}\,(\theta = 0.50)$ =0.945 and $\widehat{\phi}_{intercept}\,(\theta = 0.75)$ =1.279 for total debt, and $\phi_{intercept}\,(\theta = 0.25)$ is not statistically significant, $\widehat{\phi}_{intercept}\,(\theta = 0.50)$ =0.677 and $\widehat{\phi}_{intercept}\,(\theta = 0.75)$ =1.414 for debt held by the public.

The way growth depends on debt along the business cycle is analyzed in its steady-state

Table 3: Estimated Threshold Effects - TQARDL Models (Quarterly Data)

		θ		
		0.25	0.50	0.75
Total	τ	33.501	33.494	36.871
	$\varphi_{intercept}$			
	$\varphi_{\max(0,\triangle d_t)}$	2.489^{***} (0.477)	1.743^{***} (0.478)	
	$\varphi_{g_{t-1}}$			
	$\varphi_{g_{t-2}}$			
	$\varphi_{\triangle d_t}$			0.794^{**} (0.375)
	$\varphi_{\triangle d_{t-1}}$			-0.995^{***} (0.259)
	$\varphi_{\triangle d_{t-2}}$	-1.711^{***} (0.632)		
Public	τ	26.048	30.852	34.738
	$\varphi_{intercept}$			
	$\varphi_{\max(0,\triangle d_t)}$	1.377^{***} (0.496)		
	$\varphi_{g_{t-1}}$			-0.117^{**} (0.051)
	$\varphi_{g_{t-2}}$			
	$\varphi_{\triangle d_t}$		1.322^{***} (0.397)	0.540^{**} (0.260)
	$\varphi_{\triangle d_{t-1}}$		-1.040^{***} (0.347)	-0.512^{**} (0.241)
	$\varphi_{\triangle d_{t-2}}$			-0.581^{**} (0.223)

Notes: Total: CMPR2ext $p = 2, j = 0$ Public: CMPR2ext $p = 2, j = 1$

*,**,*** stands for statistically significant at 10%, 5%, 1% levels, respectively.

path. From the estimated models we obtain for total debt

$$\widehat{Q}_g\left(0.25|d,\triangle d\right)=\begin{cases}0.460+0.765g-1.231\triangle d, & \text{if } d\leq 33.501, \triangle d>0\\ 0.460+0.765g+1.659\triangle d, & \text{if } d\leq 33.501, \triangle d\leq 0\\ 0.460+0.765g-0.453\triangle d, & \text{if } d>33.501, \triangle d>0\\ 0.460+0.765g-0.052\triangle d, & \text{if } d>33.501, \triangle d\leq 0\end{cases}, \tag{28}$$

$$\widehat{Q}_g\left(0.50|d,\triangle d\right)=\begin{cases}0.945+0.748g-2.118\triangle d, & \text{if } d\leq 33.494, \triangle d>0\\ 0.945+0.748g, & \text{if } d\leq 33.494, \triangle d\leq 0\\ 0.945+0.748g-0.375\triangle d, & \text{if } d>33.494, \triangle d>0\\ 0.945+0.748g, & \text{if } d>33.494, \triangle d\leq 0\end{cases}, \tag{29}$$

$$\widehat{Q}_g\left(0.75|d,\triangle d\right)=\begin{cases}1.279+0.783g-0.087\triangle d, & \text{if } d\leq 36.871, \triangle d>0\\ 1.279+0.783g-0.087\triangle d, & \text{if } d\leq 36.871, \triangle d\leq 0\\ 1.279+0.783g-0.288\triangle d, & \text{if } d>36.871, \triangle d>0\\ 1.279+0.783g-0.288\triangle d, & \text{if } d>36.871, \triangle d\leq 0\end{cases}, \tag{30}$$

and for the debt held by the public

$$\widehat{Q}_g\left(0.25|d,\triangle d\right)=\begin{cases}0.282+0.791g-1.706\triangle d, & \text{if } d\leq 26.048, \triangle d>0\\ 0.282+0.791g, & \text{if } d\leq 26.048, \triangle d\leq 0\\ 0.282+0.791g-0.329\triangle d, & \text{if } d>26.048, \triangle d>0\\ 0.282+0.791g, & \text{if } d>26.048, \triangle d\leq 0\end{cases}, \tag{31}$$

$$\widehat{Q}_g\left(0.50|d,\triangle d\right)=\begin{cases}0.677+0.797g-0.470\triangle d, & \text{if } d\leq 30.852, \triangle d>0\\ 0.677+0.797g-0.470\triangle d, & \text{if } d\leq 30.852, \triangle d\leq 0\\ 0.677+0.797g-0.188\triangle d, & \text{if } d>30.852, \triangle d>0\\ 0.677+0.797g-0.188\triangle d, & \text{if } d>30.852, \triangle d\leq 0\end{cases}, \tag{32}$$

$$\widehat{Q}_g\left(0.75|d,\triangle d\right)=\begin{cases}1.414+0.797g+0.173\triangle d, & \text{if } d\leq 34.738, \triangle d>0\\ 1.414+0.797g+0.173\triangle d, & \text{if } d\leq 34.738, \triangle d\leq 0\\ 1.414+0.680g-0.380\triangle d, & \text{if } d>34.738, \triangle d>0\\ 1.414+0.680g-0.380\triangle d, & \text{if } d>34.738, \triangle d\leq 0\end{cases}. \tag{33}$$

Therefore, the estimated changes of growth, $\triangle g$, as a response of a unit change of the debt-to-GDP ratio, $|\triangle d|=1$, along the US business cycle are summarized in the following Table 4.

To a large extent, the results corroborate the findings using yearly data. The US growth is stimulated in the cases when debt (total or held by the public) as a percent of GDP decreases. Interestingly enough, an increase of total debt always penalizes growth regardless of the position

Table 4: Change of Growth Given a Unit Variation of the Debt-to-GDP Ratio (Quarterly Data)

		$\triangle g$			
	θ	$d \leq \widehat{\tau}, \triangle d = -1$	$d \leq \widehat{\tau}, \triangle d = 1$	$d > \widehat{\tau}, \triangle d = -1$	$d > \widehat{\tau}, \triangle d = 1$
Total	0.25	-7.059	-5.238	$0.221^{\#}$	-1.927
	0.50	$0^{\#}$	-8.405	$0^{\#}$	-1.488
	0.75	0.401	-0.401	1.327	-1.327
Public	0.25	$0^{\#}$	-8.163	$0^{\#}$	-1.574
	0.50	$2.315^{\#}$	$-2.315^{\#}$	0.926	-0.926
	0.75	$-0.852^{\#}$	$0.852^{\#}$	1.187	-1.187

Notes: # stands for not statistically significant

of the business cycle. The largest positive impact occurs during expansions (1.327) and the largest negative impact during recessions (-1.927). The same happens with debt held by the public.

The difference between yearly and quarterly data is that the previous conclusion holds true below and above the threshold level for yearly data whereas now with quarterly data it is more evident for total debt levels above $\widehat{\tau}$. A reason for that is the lack of observations below $\widehat{\tau}$ thus making the quantile estimation in this regime less precise (the estimates -7.059, -5.238, -8.405 are way too large in absolute value). Notice that $\widehat{\tau}$ ranges from 33% to 37% and in Figure 5 we find only a few number of total debt observations below those threshold levels.

With regard to debt held by the public we can add the following interesting conclusions: When the debt level is below its threshold, the only statistically significant (negative) impact on growth occurs during recessions after an increase of debt. For debt levels above the threshold, its (absolute) impact on growth is slightly smaller than the one found for total debt.

5 Conclusion

In this paper we discuss the effect of public debt on the US economic growth allowing it to be different for different phases of the business cycle. The dataset is the one of Reinhart and Rogoff (2010) and the model, extending Chudik et al. (2017), allows for a lagged debt-to-GDP ratio effect at the threshold component, includes interactions between the threshold variable and the growth and debt covariates, and models differences across two regimes - before and after the event of World War II.

The estimated threshold quantile ARDL regressions allow us to conclude that public authorities can adopt different policies according to where the economy is positioned in the business cycle. In steady state, less debt as a percentage of GDP stimulates the US growth, an effect that in general has been less after World War II. Moreover, when the US economy is facing a recession, reducing debt has no statistically significant impact on growth, whereas increasing it makes the economy worsen, especially if the debt-to-GDP ratio is below its threshold effect. On the contrary, if it is in a boom, only reducing debt can impact (positively) the US economy, namely whenever the debt-to-GDP ratio is again below its threshold effect. So, apparently, reducing public debt as an instrument to promote growth has lost some effectiveness during the most recent decades. Also, US policy makers are especially advised to reduce the debt-to-GDP ratio during expansions to promote growth. These findings are corroborated using quarterly data from 1966Q1 to 2019Q4 for total federal debt and federal debt held by the public as percentages of GDP.

Although this paper presents conclusions for the US economy, it would be interesting to extend the approach followed to other world economies, especially when the exercise is one of international comparison. Our TQARDL model requires long-span time series data, which will necessarily exist in the future as more data become available. As mentioned above, we have results for the UK economy but, apparently, the monotonicity assumption may not be holding true. This deserves further attention because it can be taken as evidence of model misspecification, perhaps implying the need to model the conditional quantile functions as nonlinear.

The methodology itself also deserves some further research. Firstly, there does not yet exist a statistical test for linearity in TQARDL models, in which observations have time dependence; mentioned above in a footnote. As such, we assume that threshold effects exist in our models, as has been widely demonstrated in the literature using mean regression models (see some references in the Introduction). Secondly, it is also relevant in the context of the debt-growth relationship to add and test for endogeneity in the proposed TQARDL model. The estimation of endogenous dynamic threshold quantile models is also left for future research. Finally, it would be interesting to extend the quantile regression kink design (see Chiang and Sasaki, 2019) to our proposed ARDL dynamics. Although using a regression kink model implies assuming that the regression function is continuous, it would be interesting to compare the debt-growth nexus under the threshold and kink setups. As Hansen (2017) puts it, "We conclude that the threshold test is inconclusive regarding the question of whether or not there is a regression kink effect in GDP growth due to high debt."

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