Manuel Alberto M. Ferreira. The Hahn-Banach theorem for the normed spaces. Emerging Issues in the Natural and Applied Sciences 2013; 3(1), 5-17. ***DOI:*** *10.7813/einas.2013/3-1/1*

THE HAHN-BANACH THEOREM FOR THE NORMED SPACES[1]

Prof. Dr. **Manuel Alberto M. Ferreira**

Instituto Universitário de Lisboa (ISCTE – IUL),
BRU - IUL, Lisboa **(PORTUGAL)**
manuel.ferreira@iscte.pt

DOI: 10.7813/einas.2013/3-1/1

ABSTRACT

It is intended, in this chapter, to present the Hahn-Banach theorem in its version for the normed spaces. This result is particularly important in optimization problems because of the separation theorems consequences of it.

Key words: Hahn-Banach theorem, normed spaces, separation theorems

1. THE HAHN-BANACH THEOREM

1.1. Convex Sets and Bodies

Be a real vector space *L*.

Definition 1.1.1

A set $K \subset L$ is **convex** if and only if

$$\underset{x,y \in K}{\forall} \underset{\theta \in [0,1]}{\forall} \theta x + (1-\theta) y \in K \qquad (1.1.1). \blacksquare$$

[1]This work was financially supported by FCT through the Strategic Project PEst-OE/EGE/UI0315/2011.

Definition 1.1.2

It is called **nucleus** of a set $E \subset L$, and designated $J(E)$, the set of the points $x \in E$ such that, given any $y \in L$, it is determined $\varepsilon = \varepsilon(y) > 0$ such that $x + ty \in E$ since $|t| < \varepsilon$. ■

Definition 1.1.3

A convex set with non-empty nucleus is called **convex body**.■

Theorem 1.1.1

The nucleus of any convex set *K* is also convex.

Dem.:

Suppose that $x, y \in J(K)$. Be $z = \theta x + (1-\theta)y, 0 \leq \theta \leq 1$. Then, given any $a \in L$, it is possible to determine $\epsilon_1 > 0, \epsilon_2 > 0$ such that for $|t_1| < \varepsilon_1, |t_2| < \varepsilon_2$, $x + t_1 a$ and $x + t_2 a$ belong to *K*. So, the point $\theta(x + ta) + (1-\theta)(y + ta) = z + ta$ belongs to *K* for $|t| < \varepsilon = min\{\varepsilon_1, \varepsilon_2\}$ and $z \in J(K)$.■

Theorem 1.1.2

The intersection of any convex sets family is a convex set.

Dem: Be $K = \underset{\alpha}{\cap} K_\alpha$, where each K_α is a convex set. Consider any two points *x* and *y* belonging to *K*. So $\theta x + (1-\theta)y, 0 \leq \theta \leq 1$ belongs to any K_α and, in consequence, to *K*. Then *K* is convex.■

Observation:

-The intersection of convex sets, being a convex set, is not necessarily a convex body.

Definition 1.1.4

Be *A* any part of a vector space *L*. Among the convex sets that contain *A* there is a minimal set: the intersection of the whole convex sets that contain *A*[2].This minimal convex set is called the **convex hull** of *A*.■

1.2. Homogeneous Convex Functionals

Definition 1.2.1

A functional *p* defined in *L* is **convex** if and only if

[2] There is at least one convex set that contains *A*: the space *L*.

$$\underset{x,y \in L}{\forall}\ \underset{\theta \in [0,1]}{\forall} p(\theta x + (1-\theta)y) \le \theta p(x) + (1-\theta)p(y) \qquad (1.2.1).\ \blacksquare$$

Definition 1.2.2

A functional *p* is **positively homogeneous** if and only if

$$\underset{x \in L}{\forall}\ \underset{\alpha > 0}{\forall} p(\alpha x) = \alpha p(x) \qquad (1.2.2).\ \blacksquare$$

Proposition 1.2.1

For any convex positively homogeneous functional it always holds:

i) $p(x+y) \le p(x) + p(y)$ (1.2.3),

ii) $p(0) = 0$ (1.2.4),

iii) $p(x) + p(-x) \ge 0, \underset{x \in L}{\forall}$ (1.2.5),

iv) $p(\alpha x) \ge \alpha p(x), \underset{\alpha \in \mathbb{R}}{\forall}$ (1.2.6).

Dem:

i) In fact, $p(x+y) = 2p\left(\frac{x+y}{2}\right) \le 2\left(p\left(\frac{x}{2}\right) + p\left(\frac{y}{2}\right)\right) = p(x) + p(y)$,

ii) In fact, $p(0) = p(\alpha 0) = \alpha p(0), \underset{\alpha > 0}{\forall}$. So.$p(0) = 0$,

iii) In this case, $0 = p(0) = p\big(x + (-x)\big) \le p(x)p(-x), \underset{x \in L}{\forall}$,

iv) The result is evident for $\alpha \ge 0$. For $\alpha < 0$, $0 \le p(\alpha x) + p(-\alpha x) = p(\alpha x) + p(|\alpha| x) = p(\alpha x) + |\alpha| p(x)$. That is: $p(\alpha x) \ge \alpha p(x)$. ■

1.3. Minkowsky Functionals

Definition 1.3.1

Be *L* any vector space and *A* a convex body in *L* which nucleus contains 0. **The *A* convex body Minkowsky functional**, designated $p_A(x)$, is the functional

$$p_A(x) = \inf\left\{r : \frac{x}{r} \in A, r > 0\right\} \qquad (1.3.1).\ \blacksquare$$

Theorem 1.3.1

A Minkowsky functional is convex, positively homogeneous and assumes only positive values. Reciprocally, if $p(x)$ is a convex positively homogeneous functional, assuming only positive values, and *k* appositive number, then the set

$$A = \{x: p(x) \leq k\} \qquad (1.3.2)$$

is a convex body with nucleus $\{x: p(x) < k\}$, which contains the point 0. If in (1.3.2) $k = 1$, the initial functional $p(x)$ is the *A* Minkowsky functional.

Dem: Given any element $x \in L$, $\frac{x}{r}$ belongs to *A* if *r* is great enough. Then, the number $p_A(x)$ defined by (1.3.1) is positive and finite.

But, given $t > 0$ and $y = tx$, $p_A(y) = \inf\left\{r > 0: \frac{y}{r} \in A\right\} = \inf\left\{r > 0: \frac{tx}{r} \in A\right\} = \inf\left\{tr' > 0: \frac{x}{r'} \in A\right\} = t\inf\left\{tr' > 0: \frac{x}{r'} \in A\right\} = t\, p_A(x)$.

So,

$$p_A(tx) = t\, p_A(x), \ \forall\, t > 0 \qquad (1.3.3)$$

and $p_A(x)$ is positively homogeneous.

Suppose now that $x_1, x_2 \in L$. Given any $\epsilon > 0$, choose the numbers r_i $(i = 1,2)$ in the way that $p_A(x_i) < r_i < p_A(x_i) + \epsilon$. Then $\frac{x_i}{r_i} \in A$. Then, defining $r = r_1 + r_2$, the point $\frac{x_1+x_2}{r} = \frac{r_1}{rr_1} x_1 + \frac{r_2}{rr_2} x_2$ will belong to the set of points $S = \left\{z: z = \theta\frac{x_1}{r_1} + (1-\theta)\frac{x_2}{r_2}, \forall\, \theta \in [0,1]\right\}$. As *A* is a convex set, $S \subset A$ and, in particular, $\frac{x_1+x_2}{r} \in A$. So, $p_A(x_1 + x_2) \leq r = r_1 + r_2 < p_A(x_1) +, p_A(x_2) + 2\varepsilon$. As ε is arbitrary,

$$p_A(x_1 + x_2) \leq p_A(x_1) + p_A(x_2).$$

So, $p_A(\theta x + (1-\theta)y) \leq p_A(\theta x) + p_A((1-\theta)y) = \theta p_A(x) + (1-\theta)$ $p_A(y), \forall\, x, y \in L, \forall\, \theta \in [0,1]$, since it was already shown that $p_A(x)$ is positively homogeneous.

Look now to the set defined by (1.3.2). If $x, y \in A$ and $\theta \in [0,1]$, so $p(\theta x + (1-\theta)y) \leq \theta p(x) + (1-\theta)p(y) \leq K$. In consequence, *A* is a convex set. Suppose now that $p(x) < K, t > 0$ and $y \in L$. Under these conditions, $p(x \pm ty) \leq p(x) + tp(\pm y)$. If $p(-y) = p(y) =$

$0, so\ x \pm ty \in A$ for any t. If at least one of the numbers (positive) $p(y), p(-y)$ is not null, so $x \pm ty \in A$ for

$$t < \frac{K - p(x)}{max\{p(y), p(-y)\}}.$$

From the definitions it results that p is the Minkowsky functional of the set $\{x: p(x) \leq 1\}$. ■

Observation:

-Taking in account the Theorem 1.3.1, the Minkowsky functional allows to establish a correspondence between the positively homogeneous convex functionals, assuming only positive values, and the convex bodies to which nucleus the origin belongs.

1.4. The Hahn-Banach-Theorem

Definition 1.4.1

Consider a vector space L and its subspace L_0. Suppose that in L_0 is defined a linear functional f_0. A linear functional *f* defined in the whole space *L* is an **extension** of the functional f_0 if and only if

$$f(x) = f_0(x), \underset{x \in L_0}{\forall}. ■$$

The Hahn-Banach theorem is essential in the in the resolution of the problem of finding an extension of a linear functional.

Theorem 1.4.1 (Hahn-Banach)

Be *p* a positively homogeneous convex functional defined in a real vector space *L* and L_0 an *L* subspace. If f_0 is a linear functional defined in L_0, fulfilling the condition

$$f_0(x) \leq p(x), \underset{x \in L_0}{\forall} \qquad (1.4.1),$$

so there is an extension *f* of f_0 defined in *L*, linear, and such that $f(x) \leq p(x), \underset{x \in L}{\forall}$.

Dem:

Begin showing that if $L_0 \neq L$, there is an extension of f_0, f', defined in a subspace L' such that $L \subset L'$, in order to fulfill the condition (1.4.1).

Be *z* any element of *L* not belonging to L_0; if $L'$is the subspace generated by L_0 and *z*, each element of L' is expressed in the form *tz*+*x*, being $x \in L_0$. If $f'$is an extension (linear) of the functional f_0 to L', it will happen that $f'(tz+x) = tf'(z) + f_0(x)$ or, making $f'(z) = c$,

$$f'(tz+x) = tc + f_0(x).$$

Now choose *c*, fulfilling the condition (5.1) in L', that is: in order that the inequality $f_0(x) + tc \leq p(x+tz)$, for any $x \in L_0$ and any real number *t*, is accomplished.

For $t > 0$ this inequality is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \leq p\left(\frac{x}{t} + z\right)$ or

$$c \leq p\left(\frac{x}{t} + z\right) - f_0\left(\frac{x}{t}\right) \qquad (1.4.2).$$

For $t < 0$ it is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \geq -p\left(-\frac{x}{t} - z\right)$, or

$$c \geq -p\left(-\frac{x}{t} - z\right) - f_0\left(\frac{x}{t}\right) \qquad (1.4.3).$$

Now it will be proved that there is always a number *c* satisfying simultaneously the conditions (1.4.2) and (1.4.3).

Given any two elements y' and y'' belonging to L_0,

$$-f_0(y'') + p(y''+z) \geq -f_0(y') - p(-y'-z) \qquad (1.4.4),$$

since $f_0(y'') - f_0(y') \leq p(y''-y') = p\left((y''+z) - (y'+z)\right) \leq p(y''+z) + p(-y'-z)$.

Be $c'' = \inf_{y''}\left(-f_0(y'') + p(y''+z)\right)$ and $c' = \sup_{y'}\left(-f_0(y') - p(-y'-z)\right)$. As $y'$and y'' are arbitrary, it results from (1.4.4) that $c'' \geq c'$. Choosing *c* in order that $c'' \geq c \geq c'$, it is defined the functional f' on L' through the formula

$$f'(tz + x) = tc + f_0(x).$$

This functional satisfies the condition (1.4.1). So, any functional f_0 defined in a subspace $L_0 \subset L$ and subject in L_0 to the condition (1.4.1), may be extended to a subspace L'. The extension f' satisfies the condition

$$f'(x) \leq p(x), \underset{x \in L'}{\forall}.$$

If L has an algebraic numerable base $(x_1, x_2, \dots, x_n, \dots)$ the functional in L is built by finite induction, considering the increasing sequence of subspaces

$$L^{(1)} = (L_0, x_1), L^{(2)} = \left(L^{(1)}, x_2\right), \dots$$

designating $\left(L^{(k)}, x_{k+1}\right)$ the L subspace generated by $L^{(k)}$ and x_{k+1}. In the general case, that is, when L has not an algebraic numerable base, it is mandatory to use a **transfinite induction process, for instance the Haudsdorf maximal chain theorem**.

Call $\mathcal{F}$ the set of the whole pairs$\left(L', f'\right)$, at which L' is a L subspace that contains L_0 and f' is an extension of f_0 to L' that fulfills (1.4.1). Order partially $\mathcal{F}$so that

$$\left(L', f'\right) \leq \left(L'', f''\right) \text{ if and only if } L' \subset L'' \text{and } f''_{|L'} = f'.$$

By the Haudsdorf maximal chain theorem, there is a **chain**, that is: a subset of $\mathcal{F}$ totally ordered, **maximal**, that is: not strictly contained in another chain. Call it Ω. Be Φ the family of the whole L' such that $\left(L', f'\right) \in$ Ω. Φ is totally ordered by the sets inclusion; so, the union T of the whole elements of Φ is a L subspace. If $x \in$ T then $x \in L'$ for some $L' \in$ Φ ; define $\tilde{f}(x) = f'(x)$, where $f'$is the extension of f_0 that is in the pair $\left(L', f'\right)$- the definition of $\tilde{f}$ is obviously coherent. It is easy to check that $\mathrm{T} = L$ and that $f = f'$ satisfies the condition (1.4.1).■

Now the Hahn-Banach theorem complex case, corresponding to the contribution of Hahn to the theorem, will be presented. But first:

Definition 1.4.2

A linear functional *p*, assuming only positive values, defined in a complex vector space *L*, is homogeneous convex if and only if, for any $x, y \in L$ and any complex number λ,

$$p(x+y) \leq p(x) + p(y),$$
$$p(\lambda x) = |\lambda| p(x). \blacksquare$$

Theorem 1.4.1a (Hahn-Banach)

Be *p* an homogeneous convex functional defined in a vector space *L* and f_0 a linear functional, defined in a subspace $L_0 \subset L$, fulfilling the condition

$$|f_0(x)| \leq p(x), x \in L_0.$$

Then, there is a linear functional *f* defined in *L*, satisfying the conditions

$$|f(x)| \leq p(x), x \in L; f(x) = f_0(x), x \in L_0.$$

Dem:

Call L_R and L_{0R} the real vector spaces underlying, respectively, the spaces *L* and L_0. As it is evident, *p* is an homogeneous convex functional in L_R and $f_{0R}(x) = Ref_0(x)$ a real linear functional in L_{0R} fulfilling the condition $|f_{0R}(x)| \leq p(x)$ and so,

$$f_{0R}(x) \leq p(x).$$

Then, owing to Theorem 1.4.1, there is a real linear functional f_R, defined in the whole L_R space, that satisfies the conditions

$$f_R(x) \leq p(x), x \in L_R; f_R(x) = f_{0R}(x), x \in L_{0R}.$$

But, $-f_R(x) = f_R(-x) \leq p(-x) = p(x)$, and
$|f_R(x)| \leq p(x), x \in L_R$ (1.4.5).
Define in *L* the functional *f* making

$$f(x) = f_R(x) - if_R(ix).$$

It is immediate that *f* is a complex linear functional in *L* such that

$$f(x) = f_0(x), x \in L_0; Ref(x) = f_R(x), x \in L.$$

It only misses to show that $|f(x)| \leq p(x), \underset{x \in L}{\forall}$.

Proceed by absurd. Suppose that there is $x_0 \in L$ such that $|f(x_0)| > p(x_0)$. So, $f(x_0) = \rho e^{i\varphi}, \rho > 0$, and making $y_0 = e^{-i\varphi} x_0$, it would happen that $f_R(y_0) = Re\left[e^{-i\varphi} f(x_0)\right] = \rho > p(x_0) = p(y_0)$ that is contrary to (1.4.5).■

2. THE HAHN-BANACH-THEOREM FOR THE NORMED SPACES

2.1. Normed Spaces

Definition 2.1.1

Calling *L* a vector space, a **norm** in *L* is a functional *p* such that:

-$p(x) \geq 0$,
-$p(x) = 0$ if and only if $x = 0$,
-$p(x + y) \leq p(x) + p(y)$,
-$p(\alpha x) = |\alpha| p(x)$, for every α.■

A vector space *L* with a norm is a **normed space**. It is usual to designate the norm of an element $x \in L$, $\|x\|$.

Every normed space is a metric space, with the **distance**

$$d(x, y) = \|x - y\|.$$

2.2. Continuous Linear Functionals

Be *E* a normed vector space.

Definition 2.2.1

A linear functional *f,* defined in *E,* is continuous in $x_0 \in E$ if and only if, for any $\varepsilon < 0$, there is a neighboring *U* of x_0 such that

$$|f(x) - f(x_0)| < \varepsilon \text{ for } x \in U.$$■

Definition 2.2.2

A linear functional *f,* defined in *E,* is continuous if it is continuous in all $x_0 \in E$. ■

Follow some important results on the continuity of linear functionals defined in normed vector spaces.

Proposition 2.2.1

Be *E* a normed vector space and *f* a linear functional in *E*. So

i) If *E* has finite dimension, *f* is continuous,

ii) *f* is continuous if and only if *f* is continuous at the origin,

iii) *f* is continuous if and only if *f* is bounded over the unitary ball. ■

Definition 2.2.3

Be *f* a continuous linear functional in a normed space *E*. It is called *f* **norm**, and designated $\|f\|$,

$$\|f\| = \sup_{\|x\| \le 1} |f(x)|$$

that is: **the supreme of the values that $|f(x)|$ assumes in the *E* unitary ball.** ■

Observation:

-The class of continuous linear functionals so defined, is a vector normed space, called the *E* **dual space,** designated $E^{\prime}$**.**

2.3. The Hahn-Banach Theorem Version in Normed Spaces

The Theorem 1.4.1 is as follows, in normed spaces:

Theorem 2.3.1 (Hahn-Banach)

Name *L* a subspace of a real normed space *E* and f_0 a bounded linear functional in L. So, there is a linear functional defined in *E*, extension of f_0, such that

$$\|f_0\|_{L'} = \|f\|_{E'}.$$

Dem:

It is enough to think in the functional $K\|x\|$ at which $K = \|f_0\|_{L'}$. As it is convex and positively homogeneous, it is possible to put $p(x) = K\|x\|$ and to apply Theorem 1.4.1. ■

Observation:

- To see an interesting geometric interpretation of this theorem, consider the equation $\|f_0(x)\| = 1.$ It defines, in L, an hiperplane at distance $\frac{1}{\|f_0\|}$ $of\ 0$. Considering the extension f of f_0, with norm conservation, it is obtained an hiperplane in *E*, that contains the hiperplane considered behind in *L*, and that at the same distance from the origin.

The version for normed spaces of Theorem 1.4.1a is:

Theorem 2.3.1a (Hahn-Banach)

Be E a complex normed space and f_0 a bounded linear functional defined in a subspace $L \subset E$. So, there is a bounded linear functional *f*, defined in *E*, such that

$$f(x) = f_0(x), x \in L; \|f\|_{E'} = \|f_0\|_{L'}. \blacksquare$$

2.4. Separation Theorems

In this this section, two separation theorems, important consequences of the Hahn-Banach theorem, applied to the normed vector spaces, will be presented.

Observation:

- It was seen that a convex body, in a vector space, is a convex set with non-empty nucleus. It may be stated that:

i) In a normed space, the nucleus of a set is coincident with the total of its interior points, and so

ii) In a normed space, a convex body is a convex set the has, at least, one interior point.

Theorem 2.4.1 (Separation)

Consider two convex sets *A* and *B* in a normed space *E*. If one of them, for instance *A*, has at least on interior point and $(intA) \cap B = \emptyset$, there is a continuous linear functional non-null that separates the sets *A* and *B*.■

Theorem 2.4.2 (Separation)

Consider a closed convex set A, in a normed space E, and a point $x_0 \in E$, not belonging to A. So, there is a continuous linear functional, non-null, that separates strictly $\{x_0\}$ and A.■

3. CONCLUSIONS

After a review on convex sets and bodies, homogeneous convex functionals, Minkowsky functionals and continuous convex functionals, the Hahn-Banach theorem for the normed spaces is presented, of course base on its general version. In addition two important separation theorems consequences of the Hahn-Banach theorem for the normed spaces are enounced. These last results are important in the optimization of functionals.

REFERENCES

1. J.P. Aubin. Applied Functional Analysis. John Wiley & Sons Inc., New York, 1979.
2. A.V. Balakrishnan. Applied Functional Analysis. Springer-Verlag New York Inc., New York, 1981.
3. H. Brézis. Analyse Fonctionelle (Théorie et Applications). Masson, Paris, 1983.
4. M.A.M. Ferreira. Aplicação dos Teoremas de Separação na Programação Convexa em Espaços de Hilbert. Revista de Gestão, I (2), 41-44, 1986.
5. M.A.M. Ferreira, M. Andrade and M. C. Matos. Separation Theorems in Hilbert Spaces Convex Programming. Journal of Mathematics and Technology, 1 (5), 20-27, 2010.
6. M.A.M. Ferreira and M. Andrade. Management Optimization Problems. International Journal of Academic Research, Vol. 3 (2, Part III), 647-654, 2011.
7. M.A.M. Ferreira and M. Andrade. Hahn-Banach Theorem for Normed Spaces. International Journal of Academic Research, 3 (4, Part I), 13-16, 2011.
8. M.A.M. Ferreira M and Andrade. Riesz Representation Theorem in Hilbert Spaces Separation Theorems. International Journal of Academic Research, 3 (6, II Part), 302-304, 2011.

9. M.A.M. Ferreira and M. Andrade. Separation of a Vector Space Convex Parts. International Journal of Academic Research, 4 (2), 5-8, 2012.
10. M. A. M. Ferreira, M. Andrade and J. A. Filipe. Kuhn-Tucker's Theorem for inequalities in Infinite Dimension. Journal of Mathematics and Technology, 3 (1), 57-60, 2012.
11. M.A.M. Ferreira, M. Andrade and J. A. Filipe. Weak Convergence in Hilbert Spaces. International Journal of Academic Research, 4 (4), 34-36, 2012.
12. M.A.M. Ferreira, M. Andrade, M. C. Matos, J. A. Filipe and M. Coelho. Minimax Theorem and Nash Equilibrium. International Journal of Latest Trends in Finance & Economic Sciences, 2(1), 36-40, 2012.
13. S. Kakutani. A Generalization of Brouwer's Fixed Point Theorem. Duke Mathematics Journal, 8, 1941.
14. L.V. Kantorovich and G. P. Akilov. Functional Analysis. Pergamon Press, Oxford, 1982.
15. A.N. Kolmogorov and S.V. Fomin. Elementos da Teoria das Funções e de Análise Funcional. Editora Mir, 1982.
16. M.C. Matos and M.A.M. Ferreira. Game Representation -Code Form. Lecture Notes in Economics and Mathematical Systems; 567, 321-334, 2006.
17. M.C. Matos, M.A.M. Ferreira and M. Andrade. Code Form Game. International Journal of Academic Research, 2(1), 135-141, 2010.
18. M.C. Matos, M.A.M. Ferreira, J.A. Filipe and M. Coelho. Prisonner`s Dilemma: Cooperation or Treason?. PJQM-Portuguese Journal of Quantitative Methods, Vol. 1(1), 43-52, 2010.
19. J. Nash. Non-Cooperative Games. Annals of Mathematics, 54, 1951.
20. J. von Neumann and O. Morgenstern. Theory of Games and Economic Behavior. Princeton University Press, Princeton, New Jersey, 1947.
21. J. von Neumann and O. Morgenstern. Theory of Games and Economic Behavior. John Wiley & Sons Inc., New York, 1967.
22. H.L. Royden. Real Analysis. Mac Milan Publishing Co. Inc, New York, 1968.