$M|G|\infty$ Queue Systems in the Pandemic Period Study

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Abstract

Even after the Humanity efforts and great success in infectious diseases control, still epidemics happen, being the annual influenza outbreaks examples of those occurrences. To forecast the epidemic period length is very important because, in this period, it is necessary to strengthen the health care, demanding extra availability in human and material resources, with a huge increase of expenses. More pertinently, this happens with the pandemic period, since a pandemic is an epidemic with a great population and geographical dissemination. Predominantly using results on the $M|G|\infty$ queue busy period, it is presented an application of this queue system to the pandemic period's parameters and distribution function study. The choice of the $M|G|\infty$ queue for this model is quite adequate since the greatest is the number of contagions the greatest the possibility that they occur according to a Poisson process.

1 The model

In a M|G|∞ queue system, the customers arrive according to a Poisson process at rate λ, receive, upon the arrival, a service which time length is a positive random variable with distribution function $G(.)$ and mean α, independent from the other customers' services and from the arrivals process. The traffic intensity is $\rho = \lambda\alpha$.

A pandemic is an epidemic of infectious disease that is spreading through human populations across a large region[1]. So the $M|G|\infty$ queue can be applied to the pandemic

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[1] For instance a continent, or even worldwide, see [17].

period study, owing to this system adequacy to deal with every kind of large populations[2]. Then the parameter λ is the rate at which people is infected, supposed that the infections occur according to a Poisson process. The service time is the time throughout which an infected person stays sick.

In a queue system a busy period is a period that begins when a costumer arrives at the system finding it empty, ends when a costumer abandons the system letting it empty and in it there is always at least one customer present. So in a queuing system there is a sequence of idle and busy periods.

In the M|G|∞ queue system the idle periods have an exponential time length with mean λ^{-1}, as it happens with any queue system with Poisson arrivals. Although the busy period's distribution is much more complicated it is possible to present some results as it will be seen.

For what interests in this work

-A busy period is a pandemic period

-An idle period is a period free of the disease.

Another work on this subject is[8].

2 The Pandemic Period

Call PP the random variable pandemic period length. According to the results known

for the $M|G|\infty$ queue busy period length distribution

$$E[PP] = \frac{e^\rho - 1}{\lambda} \qquad (2.1)$$

whichever is an infected person sickness time length distribution, see [16]. As for $Var[PP]$, it depends on the whole sickness time length distribution probabilistic structure. But Sathe, see [15], demonstrated that

$$\lambda^{-2}\max[e^{2\rho} + e^\rho \rho^2 \gamma_s^2 - 2\rho e^\rho - 1; 0] \leq Var[PP] \leq \lambda^{-2}$$

$$[2e^\rho(\gamma_s^2 + 1)(e^\rho - 1 - \rho) - (e^\rho - 1)^2] \qquad (2.2),$$

where γ_s is the sickness time length coefficient of variation. If an infected person sickness time length distribution function is

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[2]For more examples of the $M|G|\infty$ queue practical applications, see, for instance, [3], [5], [10 − 14]and [16] .

$$G(t) = \frac{e^{-\rho}}{(1 - e^{-\rho})e^{-\lambda t} + e^{-\rho}}, t \geq 0, \quad (2.3)$$

the *PP* distribution function is

$$PP(t) = 1 - (1 - e^{-\rho})e^{-e^{-\rho}\lambda t}, t \geq 0 \quad (2.4),$$

see [2]. If the sickness time length distribution function of an infected person is such that

$$G(t) = 1 - \frac{1}{1 - e^{-\rho} + e^{-\rho + \frac{\lambda}{1 - e^{-\rho}}t}}, t \geq 0 \quad (2.5)$$

the *PP* distribution function is

$$PP(t) = 1 - e^{-(e^{\rho} - 1)^{-1}\lambda t}, t \geq 0 \quad (2.6),$$

see [4]. For α and ρ great enough (very intense infectious conditions) since $G(.)$ is such that for α great enough $G(t) \cong 0, t \geq 0$,

$$PP(t) \cong 1 - e^{-\lambda e^{-\rho}t}, t \geq 0 \quad (2.7),$$

see [12].

Note:

-As for this last result, begin noting that many probability distributions fulfill the condition $G(t) \cong 0, t \geq 0$ for α great enough. The exponential distribution is one example.

-As for the meaning of α and ρ great enough, through computations presented in [14] it is shown that for $\lambda = 1$, after $\rho = 10$ it is reasonable to admit (2.7) for many service time distributions.∎

Calling N_{PP} the mean number of sick people in the pandemic period, if $G(.)$ is exponential

$$N_{PP} = e^{\rho} \quad (2.8).$$

For any other $G(.)$ probability distribution

$$N_{PP} \cong \frac{e^{\rho(\gamma_s^2 + 1)}(\rho(\gamma_s^2 + 1) + 1) + \rho(\gamma_s^2 + 1) - 1}{2\rho(\gamma_s^2 + 1)} \quad (2.9),$$

see [10]. Of course, multiplying (2.8) or (2.9), as appropriate, by the mean cost of each sick person treatment it is possible to estimate the health care costs caused by the pandemic period.

Be $p_{1'0}(t)$ the probability that there is no seek people at time t, being the time origin the pandemic period beginning. Being $h(t) = \frac{g(t)}{1-G(t)}$ where $g(t)$ is the probability density function associated to $G(.)$, the service time hazard rate function[3],

$$h(t) \geq \lambda \Rightarrow p_{1'0}(t) \text{ is non} - \text{decreasing} \quad (2.10),$$

see Proposition 3.1 in [9]. And calling $\mu(1', t)$ the mean number of seek people at time t, being the time origin the pandemic period beginning

$$h(t) \leq \lambda \Rightarrow \mu(1', t) \text{ is non} - \text{decreasing} \quad (2.11),$$

see Proposition 5.1 in [9].

3 Number of Pandemic Periods occurrence in a Time Interval

After the renewal processes theory, see [1], calling $R(t)$ the mean number of pandemic periods that begin in $[0, t]$, being $t = 0$ the beginning instant of a pandemic period, it is possible to obtain, see [6,7],

$$R(t) = e^{-\lambda \int_0^t [1-G(v)]dv} + \lambda \int_0^t e^{-\lambda \int_0^u [1-G(v)]dv} du \quad (3.1)$$

and, consequently,

$$e^{-\rho}(1 + \lambda t) \leq R(t) \leq 1 + \lambda t \quad (3.2),$$

see [6]. Also,

A) $G(t) = \frac{e^{-\rho}}{(1-e^{-\rho})e^{-\lambda t}+e^{-\rho}}, t \geq 0$

$$R(t) = 1 + \lambda e^{-\rho} t \quad (3.3)$$

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[3]That is: the rate at which sick people is free of the disease.

B) $G(t) = 1 - \dfrac{1}{1 - e^{-\rho} + e^{-\rho + \frac{\lambda}{1 - e^{-\rho}}t}}, t \geq 0$

$$R(t) = e^{-\rho} + (1 - e^{-\rho})^2 + \lambda e^{-\rho}t + e^{-\rho}(1 - e^{-\rho})e^{-\frac{\lambda}{1 - e^{-\rho}}t} \quad (3.4)$$

C) $G(t) = \begin{cases} 0, t < \alpha \\ 1, t \geq \alpha \end{cases}$

$$R(t) = \begin{cases} 1, t < \alpha \\ 1 + \lambda e^{-\rho}(t - \alpha), t \geq \alpha \end{cases} \quad (3.5)$$

D) If the sickness time length is exponentially distributed

$$e^{-\rho\left(1 - e^{-\frac{t}{\alpha}}\right)} + \lambda e^{-\rho}t \leq R(t) \leq e^{-\rho\left(1 - e^{-\frac{t}{\alpha}}\right)} + \lambda t \quad (3.6)$$

4 Conclusions

So that this model can be applied it is necessary that the infections occur according to a Poisson process at constant rate. It is a hypothesis perfectly admissible in this kind of phenomena, since they have great geographic spread, even worldwide, and that it is considered the mean arrival rate for the pandemic period as the constant rate, because the rate at which infections occur is evidently variable along the pandemic period.

Among the results presented, (2.1), (2.2), (2.7) and (3.2) are remarkable for the easiness and also for requiring only the knowledge of the infectious rate λ, the mean sickness time α, and the sickness time variance. The other results are more complex and demand the goodness of fit test for the distributions indicated to the sickness times.

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