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PROCEEDINGS

THE CONCEPT OF WEAK CONVERGENCE IN HILBERT SPACES

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Abstract. In order to generalize the Bolzano-Weierstrass Theorem, a weaker notion of convergence is introduced. Then it is discussed in which conditions weak convergence implies convergence. The results presented are in the domain of the real Hilbert spaces.

Keywords: Hilbert spaces, Bolzano-Weierstrass Theorem, weak convergence, convergence

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1 Introduction

Through the Bolzano-Weierstrass Theorem it is established that a bounded sequence of real numbers has at least one sublimit. This result remains true for any finite dimension space with inner product, that is in $\mathbb{R}^n$.
A result like this does not happen when infinite dimension spaces are considered. In fact, under those conditions it is possible to find a sequence of terms in a Hilbert space H, ortonormal, designated $\{h_n\}$. So $\|h_n\| = 1$ and $\|h_n - h_m\|^2 = [h_n - h_m, h_n - h_m] = \|h_n\|^2 + \|h_m\|^2 = 1 + 1 = 2$, if $m \neq n$[1].
Consequently this sequence is bounded and has not sublimits.
Then it is legitimate to ask which is the generalization of the Bolzano-Weierstrass Theorem?

2 Introducing Weak Convergence

Note that for any $g \in H$, and for the ortonormal sequence seen above, $\|g\|^2 \geq \sum_{k=1}^{\infty} |[g, h_k]|^2$, according to Bessel's inequality. In consequence

$$\lim_k [g, h_k] = 0 = [g, 0], \forall g \in H.$$

Based on this example, a weaker notion of convergence will be introduced.

Definition 2.1
A sequence x_k of elements in H converges weakly for an element x belonging to H if and only if $\lim_k [x_k, g] = [x, g]$ for any g in H.

[1] $[.,.]$ means inner product and $\|.\|$ means norm.

Definition 2.2
An element y is a weak limit of a set M if and only if $[x, y]$ is a limit point of $[x, M]$ for any x in H.

Definition 2.3
A set M is weakly closed if and only if contains all its weak limits.

Observation:
Every set weakly closed is closed. The reciprocal proposition is not true.

Now two theorems at which important properties for the Hilbert spaces are established will be enounced without demonstration. The second is true in any Banach space. To demonstrate the first it would be necessary, in particular, the Riez Representation Theorem, see (Ferreira and Andrade, 2011b). For the second it would be necessary the Baire Category Theorem, see (Royden, 1968), true for any complete metric space.

Theorem 2.1 (Weak Compactness Property)
Every bounded sequence of elements in a Hilbert space contains at least a subsequence weakly convergent.

Theorem 2.2 (Uniform Boundary Principle)
Be $f_n(.)$ a sequence of continuous linear functionals in H such that $\sup_n |f_n(x)| < \infty$ for each x in H. Then $\|f_n(.)\| \leq M$ for any $M < \infty$.

Two corollaries, very useful, from this theorem are:
Corollary 2.1
Be $f_n(.)$ a sequence of continuous linear functionals such that, for each $x \in H, f_n(x)$ converges. Then there is a continuous linear functional such that $f(x) = lim\, f_n(x)$ and $\|f(.)\| \leq \underline{lim}\, \|f_n(.)\|$.

Dem:
By the Uniform Boundary Principle, it follows that $\|f_n(.)\| \leq M$ for any $M < \infty$. Define $g(x) = lim f_n(x)$. So $g(.)$ is evidently linear. Suppose that $\|x_m - x\| \to 0$. So $|g(x_m - x)| = \lim_n |f_n(x_m - x)| \leq M\|x_m - x\| \to 0$. Consequently $g(.)$ is continuous. Also for any x, $\|x\| = 1, |g(x)| = lim|f_n(x)| \leq \underline{lim}\, \|f_n(.)\|$.■

Corollary 2.2
Be $f_n(.)$ a sequence of continuous linear functionals such that $\|f_n(.)\| \leq M$ and $f_n(.)$
converges for each x in a dense subset of H. Then,

- There is a linear continuous functional $f(.)$ such that $\lim_n f_n(x) = f(x)$ since this limit exists,
- The limit linear functional is unique.

Dem:
It will be stated that $f_n(x)$, in fact, converges for every x in H. For it, be x_n in the dense set[2]:

$$\|x - x_n\| \to 0;\ f_m(x_n) \text{ converges in } m.$$

Consider p, great enough such that, given $\varepsilon > 0, \|x - x_p\| \leq \frac{\varepsilon}{4M}$.

[2] That is: be x_n, elements of the dense set, such that $x_n \to x$.

Consider also n and m so that $|f_n(x_p) - f_m(x_p)| \leq \frac{\varepsilon}{2}$. Then

$$|f_m(x) - f_n(x)| \leq |f_m(x - x_p) - f_n(x - x_p)| + |f_m(x_p) - f_n(x_p)| \leq 2M\|x - x_p\| + \frac{\varepsilon}{2} \leq$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then, $f_n(x)$ converges and the conditions of the former corollary are fulfilled. ■

3 Weak Convergence and Convergence

By curiosity, it is natural to pose the following question:

- Under which conditions weak convergence implies convergence?

The first result important to answer this question is:

Theorem 3.1
Suppose that x_n converges weakly for x and $\|x_n\|$ for $\|x\|$. Then x_n converges for x.

Dem:
It is immediate that

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - [x_n, x] - [x, x_n] \to \|x\|^2 + \|x\|^2 - 2[x, x] = 2\|x\|^2 - 2\|x\|^2 = 0.$$

Consequently $\|x_n - x\|^2 \to 0$.■

Much more useful than the former one in the applications, on weak convergence, is the following result due to Banach-Sacks:

Theorem 3.2 (Banach-Sacks)
Suppose that x_n converges weakly for x. Then it is possible to determine a subsequence $\{x_{n_k}\}$ such that the arithmetical means $\frac{1}{m}\sum_{k=1}^{m} x_{n_k}$ converge for x.

Dem:
Generality lossless, it may be supposed that $x = 0$. Consider x_{n_k} as follows:

- $x_{n_1} = x_1$,
- Due to the weak convergence, it is possible to choose x_{n_2}, such that $|[x_{n_1}, x_{n_2}]| < 1$,
- Having considered x_{n_1}, …, x_{n_k} it is evident that it is admissible to choose $x_{n_{k+1}}$ such that $|[x_{n_i}, x_{n_{k+1}}]| < \frac{1}{k}, i = 1, 2, \dots, k$.

As, by the uniform boundary, it is possible to take $\|x_{n_k}\| \leq M$ for any $M < \infty$, with the inner products usual calculations rules it is obtained:

$$\left\|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}\right\|^2 \leq \left(\frac{1}{k}\right)^2 \left(kM + 2\sum_{i=2}^{k}\sum_{j=1}^{i-1} |[x_{n_j}, x_{n_i}]|\right) \leq \frac{1}{k^2}\left(kM + 2(k-1)\right) \to 0.$$

So $\frac{1}{m}\sum_{k=1}^{m} x_{n_k}$ converges to 0. ■

Observation:
An alternative formulation of Theorem 3.2 is:

- Every closed convex subset is weakly closed.

Finally a Corollary of Theorem 3.2.

Corollary 3.1 (Convex Functionals Weak Inferior Semicontinuity)
Be $f(.)$ a continuous convex functional in the Hilbert space *H*. So if x_n converges weakly to *x*, $\underline{lim}\, f(x_n) \geq f(x)$.

Dem:
Consider a subsequence x_{n_m}, and put $x_m = x_{n_m}$, in order that $\underline{lim}\, f(x_n) = \lim\, f(x_m)$ and, still, that $\frac{1}{n}\sum_{m=1}^{n} x_m$ converges for *x*, in accordance with Theorem 3.2. But, as $f(.)$ is convex,

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right).$$

So, $\underline{lim}\, f(x_n) = \lim \frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq \lim f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) = f\,(x)$.■

4 Conclusions

The notion of weak convergence established in Definition 2.1 allows a possible generalization of Bolzano-Weierstrass Theorem. The Theorem 2.1 (Weak Compactness Property) and the Theorem 2.2 (Uniform Boundary Principle) help to understand that notion. And the Corollary 2.1 and the Corollary 2.2 to establish some operational properties. Finally with the help of Banach-Sacks Theorem are presented conditions under which weak convergence implies convergence.

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