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DIFFERENTIAL EQUATIONS IMPORTANT IN THE M|G|∞ QUEUE SYSTEM TRANSIENT BEHAVIOUR AND BUSY PERIOD STUDY

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Abstract. This paper main subject is the M|G|∞ queue system populational process transient probabilities study as time functions. We achieve it completely when the time origin is an unoccupied system instant. But we do not get such a goal when the time origin is a busy period beginning instant. We shall see that, in this last situation, the service time length distribution hazard rate function plays a very important role. And so the results got may be useful in the survival analysis field. As the M|G|∞ queue can be applied in the modelation of many social problems: sickness, unemployment, emigration, ...(see, for instance, Ferreira (1995, 1996, 2003a and 2003b)), in these situations it is very important to study the busy period length distribution of that system. We show, in this work, that the solution of the problem may be in the resolution of a Ricatti equation generalizing the work of Ferreira (1998 and 2003) as a consequence of the transient behaviour study. We put also a special incidence in the study of the mean and the variance of the transient probabilities when the time origin is a busy period beginning instant, as time function. It allows the consideration of very interesting linear differential equations.

Key words: M|G|∞, transient behaviour, hazard rate function, busy period, Ricatti equation, first order differential equation

1 Introduction

We call M|G|∞ a queue system at which the customers arrive according to a Poisson process at rate λ, receive a service whose time length is a positive random variable with distribution function $G(.)$ and mean α. So $\alpha=\int_0^\infty \left[1-G(v)\right]dv$. Upon its arrival each customer finds immediately a server available. Each customer service is independent from the other customers services and from the arrival process. The traffic intensity is $\rho = \lambda\alpha$.

In a M|G|∞ queue there are infinite servers. But in general we do not impose the physical presence of infinite servers. Instead, in practical applications, we suppose that when someone arrives at the system it finds immediately a server available. Or that there is no distinction between a customer and its server. This is the key idea to use the M|G|∞ queue to model sickness and unemployment problems, for example.

Being $N(t)$ the occupied servers number, that is the same that the being served customers number, at time t in a M|G|∞ system, according to Carrillo (1991) we have, putting

$$p_{0n}(t) = P\left[\, N(t) = n \middle| N(0) = 0 \,\right],\ n = 0,1,2,...,\ \text{that}$$

$$p_{0n}(t) = \frac{\left(\lambda \int_0^t \left[\, 1 - G(v) \,\right] dv\right)^n}{n!} e^{-\lambda \int_0^t \left[\, 1 - G(v) \,\right] dv}, n = 0,1,2,... \qquad (1.1).$$

So, the transient distribution, being the time origin an empty system instant, is Poisson with mean

$$\lambda \int_0^t \left[\, 1 - G(v) \,\right] dv.$$

The stationary distribution is the limit one:

$$\lim_{t \to \infty} p_{0n}(t) = \frac{\rho^n}{n!} e^{-\rho}, n = 0,1,2,...$$

being Poisson with mean ρ.

In the M|G|∞ system, as in any other queue system, operation there is a sequence of idle and busy periods. An idle period followed by a busy period is what we call a busy cycle. In sickness applications, an idle period is a period where nobody is sick. A busy period is a period that corresponds to an epidemic. A busy cycle is a sickness cycle. In unemployment applications, an idle period is a full employment period. A busy period is a period of unemployment. A busy cycle is an employment cycle.

So it is important to study the M|G|∞ busy cycle length distribution and not only the busy period length distribution. The M|G|∞ idle period length is exponential with parameter λ, as it happens with any queue system busy period with Poisson arrival process at rate λ.

Be $p_{1'n}(t) = P[N(t) = n | N(0) = 1'], n = 0,1,2,...,$ meaning $N(0) = 1'$ that the time origin is the one of a customer arrival at the system, making the number of served customers jump from 0 to 1. That is: a busy period begins.

At $t \geq 0$, possibly:

- The customer that arrived at the time origin abandoned the system, with probability $G(t)$, or goes on being served, with probability $1 - G(t)$;
- The other servers, that were unoccupied at the time origin, are still unoccupied or occupied with 1,2,... customers, with probabilities given by $p_{0n}(t), n = 1,2,...$

Both subsystems, the one of the initial customer and the other of the initially unoccupied servers, are independent and so we have

$$p_{1'0}(t) = p_{00}(t)G(t) \qquad (1.2).$$
$$p_{1'n}(t) = p_{0n}(t)G(t) + p_{0n-1}(t)(1 - G(t)), \qquad n = 1,2,...$$

For the M | M | ∞ system (exponential service times) (1.2) is valid even if $N(0)=1$, that is: since the time origin in an instant at which there is one only customer in the system, simply, owing to the exponential distribution lack of memory. So

$$p_{10}^{M}(t)=\left(1-e^{-\frac{t}{\alpha}}\right)e^{-\rho\left(1-e^{-\frac{t}{\alpha}}\right)}$$

$$p_{1n}^{M}(t)=\frac{1}{(n-1)!}\rho^{n-1}\left(1-e^{-\frac{t}{\alpha}}\right)^{n-1}e^{-\rho\left(1-e^{-t/\alpha}\right)}\left(\frac{\left(1-e^{-\frac{t}{\alpha}}\right)^{2}\rho}{n}+e^{-\frac{t}{\alpha}}\right),\qquad n=1,2,\ldots$$

It is easy to show that

$$\lim_{t\to\infty}p_{1'n}(t)=\frac{\rho^{n}}{n!}e^{-\rho},n=0,1,2,\ldots$$

Calling $\mu(1',t)$ and $\mu(0,t)$ the mean values associated to the distributions given by (1.2) and (1.1), respectively, we have that

$$\mu(1',t)=\sum_{n=1}^{\infty}np_{1'n}(t)=\sum_{n=1}^{\infty}nG(t)p_{0n}(t)+\sum_{n=1}^{\infty}np_{0n-1}(t)(1-G(t))=$$

$$=G(t)\mu(0,t)+(1-G(t))\sum_{j=0}^{\infty}(j+1)p_{0j}(t)=\mu(0,t)+(1-G(t)).$$

So

$$\mu(1',t)=1-G(t)+\lambda\int_{0}^{t}\left[1-G(v)\right]dv \qquad (1.3).$$

As

$$\sum_{n=0}^{\infty}n^{2}p_{0n}(t)=G(t)\sum_{n=1}^{\infty}n^{2}p_{0n}(t)+(1-G(t))\sum_{n=1}^{\infty}n^{2}p_{0n-1}(t)=$$

$=G(t)\left(\mu^{2}(0,t)+\mu(0,t)\right)+(1-G(t))\left(\mu^{2}(0,t)+3\mu(0,t)+1\right)=\mu^{2}(0,t)+(3-2G(t))\mu(0,t)+1-G(t)$, designated by $V(1',t)$ the variance associated to the distribution given by (1.2), we obtain

$$V(1',t)=\mu(0,t)+G(t)-G^{2}(t) \qquad (1.4).$$

We intend to present some results about $p_{0n}(t), n=0,1,2,\ldots$, $p_{1'0}(t)$, $\mu(1',t)$ and $V(1',t)$ behaviours as time functions. We will show too that the $p_{1'0}(t)$ study induces a Ricatti equation important to the determination of a M | G | ∞ systems collection with practically exponential busy period. $\mu(1',t)$ and $V(1',t)$ studies induce also differential equations that allow very interesting results.
We will see that, in its behaviour as time functions, takes place the service hazard rate function with an important role given by, see for example (Ross, 1983),

$$h(t)=\frac{g(t)}{1-G(t)} \quad (1.5)$$

where $g(.)$ is the density associated to $G(.)$.

2 $p_{0n}(t)$, $n = 0,1,2,...$ BEHAVIOUR AS TIME FUNCTION

This section main result is:

Proposition 2.1

If $G(t)<1$, $t>0$, continuous and differentiable

a) $p_{00}(t)$, $t>0$ is a decreasing function,

b) $p_{0n}(t)$, $n \geq \rho$, $t>0$ is an increasing function,

c) $p_{0n}(t)$, $0<n<\rho$, $\rho>1$

i) increases in $]0,t_n[$ being t_n given by

$$\int_0^{t_n}[1-G(v)]dv = \frac{n}{\lambda} \tag{2.1}$$

ii) decreases in $]t_n, \infty[$ and

iii)the $p_{0n}(t)$ maximum is

$$p_{0n}(t_n) = \frac{n^n}{n!}e^{-n} \tag{2.2}$$

Dem: a) is evident since

$$p_{00}(t) = e^{-\lambda\int_0^t[1-G(v)]dv}.$$

For $n \geq 1$ $\frac{d}{dt}p_{0n}(t) = \lambda p_{0n}(t)(1-G(t))\left(\frac{n}{\lambda\int_0^t[1-G(v)]dv}-1\right), t>0$.

As $\lambda\int_0^t[1-G(v)]dv < \rho$, if $n \geq \rho, \frac{d}{dt}p_{0n}(t) > 0, t>0$ and we conclude b).

If

$$n<\rho, \frac{d}{dt}p_{0n}(t) = 0 \Leftrightarrow \int_0^t[1-G(v)]dv = \frac{n}{\lambda} \text{ and we have c).}$$

Notes:

Although t_n, given by (2.1), depends on the arrival rate and on the service time length distribution, that does not happen with $p_{0n}(t_n)$ given by (2.2),

For a certain arrival rate and service time length distribution we have, evidently,

$$t_{n+1} \geq t_n$$

and, as

$$\frac{p_{0n+1}(t_{n+1})}{p_{0n}(t_n)} = \frac{(n+1)^{n+1}}{(n+1)!}e^{-n-1}\frac{n!}{n^n}e^n = (n+1)\left(\frac{n+1}{n}\right)^n\frac{e^{-1}}{n+1} =$$

$$= \left(1+\frac{1}{n}\right)e^{-1} \le ee^{-1} = 1, \qquad p_{0n+1}(t_{n+1}) \le p_{0n}(t_n)$$

Under Proposition (2.1) conditions, but with $1-G(t)=0, t \ge t_l$, (1.1) becomes

$$p_{0n}(t) = \frac{\left(\lambda\int_0^t\left[1-G(v)\right]dv\right)^n}{n!}e^{-\lambda\int_0^t\left[1-G(v)\right]dv}, \quad t \le t_l \quad \text{and} \quad p_{0n}(t) = \frac{\rho^n}{n!}e^{-\rho}, t > t_l,$$

$, n = 0,1,2,...$

and, so, the Proposition (2.1) conclusions are still valid, but the values $\frac{\rho^n}{n!}e^{-\rho}, n = 0,1,2,...$ occur after $t = t_l$. Evidently, $t_n < t_l, 0 < n < \rho, \rho > 1$.

3 $p_{1'0}(t)$ BEHAVIOUR AS TIME FUNCTION

For the $p_{1'n}(t), n = 0,1,2,...$ it is not possible to perform such a complete study as for the $p_{0n}(t), n = 0,1,2,...$. But the results for $p_{1'0}(t)$ are very interesting as we will see. Now the important result is

Preposition 3.1

If $G(t) < 1, t > 0$, continuous, differentiable and

$$h(t) \ge \lambda G(t), t > 0 \qquad (3.1),$$

being $h(t)$ the hazard rate function associated to $G()$, $p_{1'0}(t)$ is non-decreasing.
Dem: It is enough to note that, under these conditions,

$$\frac{d}{dt}p_{1'0}(t) = p_{00}(t)(1-G(t))\left(\frac{g(t)}{1-G(t)} - \lambda G(t)\right) \text{ where } g(t) = \frac{d}{dt}G(t) \text{ and } h(t) = \frac{g(t)}{1-G(t)}.$$

Notes:

- Note that
$$h(t) \ge \lambda \qquad (3.2)$$
is a sufficient condition for (3.1) and so if the rate at which the services end is greater or equal than the customers arrival rate $p_{1'0}(t)$ is non-decreasing,
- For the M|M|∞ system (3.2) is equivalent to

$$\rho \leq 1$$

- Evidently these results may be useful in the survival analysis fields.

Putting

$$h(t) - \lambda G(t) = \beta, \beta \in IR$$

we have a Ricatti equation whose solution is (note that $G(t) = 1,\ t \geq 0$ is a solution)

$$G(t) = 1 - \frac{\left(1 - e^{-\rho}\right)(\lambda + \beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t} - 1\right) + \lambda}, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho} - 1} \quad (3.3),$$

see Ferreira (1998). For a $\mathrm{M}|\mathrm{G}|\infty$ system with this service time length distribution

$$p_{1'0}(t) = e^{-\rho} - \frac{\left(1 - e^{-\rho}\right)\beta}{\lambda} e^{-(\lambda+\beta)t}, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho} - 1}.$$

Concretely

- $\beta = -\lambda$ we get

$$p_{1'0}(t) = 1, t \geq 0.$$

  In fact, in this situation, $G(t) = 1, t \geq 0$. So $G(.)$ is degenerated at the origin. That is: every customer has null service time length. So the system is never occupied,

  $\beta = 0$

$$p_{1'0}(t) = e^{-\rho}, t \geq 0$$

  and so $p_{1'0}(t), t \geq 0$ is constant,

- $\beta = \dfrac{\lambda}{e^{\rho} - 1}$

$$p_{1'0}(t) = e^{-\rho}\left(1 - e^{-(\lambda+\beta)t}\right), t \geq 0.$$

With the service time length distribution given by (3.3), (1.1) becomes

$$p_{0n}(t) = \frac{\left(-\log\left(e^{-\rho} + \left(1 - e^{-\rho}\right)e^{-(\lambda+\beta)t}\right)\right)^{n}}{n!}\left(e^{-\rho} + \left(1 - e^{-\rho}\right)e^{-(\lambda+\beta)t}\right),$$

$$t \geq 0, -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho} - 1}.$$

Calling T the random variable associated to $G(.)$ given by (3.3) we have, see Ferreira (1998 a)

$$\frac{\left(1 - e^{-\rho}\right)e^{-\rho}}{\lambda} \frac{n!}{(\lambda + \beta)^{n-1}} \leq E\left[T^{n}\right] \leq \frac{e^{\rho} - 1}{\lambda} \frac{n!}{(\lambda + \beta)^{n-1}},$$

$$-\lambda < \beta \leq \frac{\lambda}{e^{\rho}-1}, n = 1,2,... \tag{3.4}$$

Notes:

- The expression (3.4) giving bounds for $E\left[T^{n}\right], n = 1,2,...$ proves its existence,
- For $n = 1$ (3.4) is unuseful because we know that $E[T] = \alpha$. Curiously, the upper bound is $\frac{e^{\rho}-1}{\lambda}$, the M | G | ∞ system busy period length mean value,
- For $\beta = -\lambda, E\left[T^{n}\right] = 0, n = 1,2,...$ evidently.

See however that (3.3) may take the form

$$G(t) = \frac{1+\frac{\beta}{\lambda}\left(1-e^{\rho}\right) e^{-(\lambda+\beta)t}}{1-\left(1-e^{\rho}\right) e^{-(\lambda+\beta)t}}, \quad t \geq 0, \quad -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho}-1}$$

and, since $\rho < \log 2$,

$$G(t) = \left(1+\frac{\beta}{\lambda}\left(1-e^{\rho}\right) e^{-(\lambda+\beta)t}\right)\sum_{k=0}^{\infty}\left(1-e^{\rho}\right)^{k} e^{-k(\lambda+\beta)t},$$

$$t \geq 0, -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho}-1} \tag{3.5}$$

After (3.5) we can easily compute the T Laplace transform for $\rho < \log 2$. And then get

$$E\left[T^{n}\right] = -\left(1+\frac{\beta}{\lambda}\right) n! \sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{\left(k\left(\lambda+\beta\right)\right)^{n}}, -\lambda < \beta \leq \frac{\lambda}{e^{\rho}-1}, \rho < \log 2, n = 1,2,...$$

Notes:

$$E[T] = -\left(1+\frac{\beta}{\lambda}\right)\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{k\left(\lambda+\beta\right)} = -\frac{\lambda+\beta}{\lambda\left(\lambda+\beta\right)}.\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{k} =$$

$$= \frac{1}{\lambda}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(e^{\rho}-1\right)^{k}}{k} = \frac{1}{\lambda}\log e^{\rho} = \frac{\rho}{\lambda} = \alpha .$$

For $n \geq 2$ we must truncate the infinite sum. Taking only M terms, to get an error lesser or equal than ε we must have simultaneously

$$M > \frac{1}{\lambda+\beta} - 1$$

$$M > \log_{\left(e^{\rho}-1\right)}\frac{e^{\rho}\lambda}{n!\left(\lambda+\beta\right)} - 1$$

4 A M|G|∞ SYSTEMS COLLECTION WITH EXPONENTIAL BUSY PERIOD

Putting now $h(t)-\lambda G(t)=\beta(t)$ ($\beta(.)$ is any time function) we get

$$\frac{dG(t)}{dt}=-\lambda G^2(t)-(\beta(t)-\lambda)G(t)+\beta(t)$$

that is also a Ricatti equation about $G(.)$.

Solving it, after noting that $G(t)=1, t\geq 0$ is a solution again, we get

$$G(t)=1-\frac{1}{\lambda}\frac{\left(1-e^{-\rho}\right)\ e^{-\lambda t-\int_0^t \beta(u)du}}{\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw-\left(1-e^{-\rho}\right)\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw},$$

$$t\geq 0,-\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^\rho-1} \qquad (4.1).$$

Putting (4.1) in

$$\overline{B}(s)=1+\lambda^{-1}\left(s-\frac{1}{\int_0^\infty e^{-st-\lambda\int_0^t[1-G(v)]dv}dt}\right)$$

that is the M|G|∞ busy period length Laplace transform, see Stadje (1985), we get

$$\overline{B}(s)=\frac{1-(s+\lambda)\left(1-G(0)\right)L\left[e^{-\lambda t-\int_0^t \beta(u)du}\right]}{1-\lambda\left(1-G(0)\right)L\left[e^{-\lambda t-\int_0^t \beta(u)du}\right]},-\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^\rho-1} \qquad (4.2)$$

where L means Laplace transform and

$$G(0)=\frac{\lambda\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw+e^{-\rho}-1}{\lambda\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw}.$$

After (4.2) we can compute $\frac{1}{s}\overline{B}(s)$ whose inversion gives

$$B(t)=\left(1-(1-G(0))\left(e^{-\lambda t-\int_0^t \beta(u)du}+\lambda\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw\right)\right)*$$

$$*\sum_{n=0}^\infty \lambda^n(1-G(0))^n\left(e^{-\lambda t-\int_0^t \beta(u)du}\right)^{*n},-\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^\rho-1} \qquad (4.3)$$

for the M|G|∞ busy period d.f., where $*$ is the convolution operator.
If $\beta(t)=\beta$ (constant), we get (3.3) and

$$B^{\beta}(t)=1-\frac{\lambda+\beta}{\lambda}\left(1-e^{-\rho}\right)e^{-e^{-\rho}(\lambda+\beta)t}, t\geq 0, -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1}$$

So, if the service time d.f. is given by (3.3) the M|G|∞ busy period d.f. is the a mixture of a degenerate distribution at the origin and an exponential distribution.

Finally note that, for $\beta=\frac{\lambda}{e^{\rho}-1}$, $B^{\beta}(t)=1-e^{-\frac{\lambda}{e^{\rho}-1}t}$, $t\geq 0$ (purely exponential). And $B(t)$, given by (4.3) satisfies

$$B(t)\geq 1-e^{-\frac{\lambda}{e^{\rho}-1}t},\ t\geq 0,\ -\lambda\leq\frac{\int_0^t\beta(u)du}{t}\leq\frac{\lambda}{e^{\rho}-1}$$

Calling the busy cycle length Laplace trasform $\overline{Z(s)}$, knowing that the M|G|∞ idle period is exponential with parameter λ, as it happens with any queue system with Poisson arrival at rate λ, and that the idle period and the busy period are independent for the M|G|∞ queue (Takács (1962)) we have

$$\overline{Z(s)}=\frac{\lambda}{\lambda+s}\overline{B(s)} \tag{4.4}$$

After (4.2) and (4.4) we can compute $\frac{1}{s}\overline{Z}(s)$ whose inversion gives

$$Z(t)=\left(\lambda e^{-\lambda t}\right)*\left(1-\left(1-G(0)\right)\left(e^{-\lambda t-\int_0^t\beta(u)du}+\lambda\int_0^t e^{-\lambda w-\int_0^w\beta(u)du}dw\right)\right)*$$

$$*\sum_{n=0}^{\infty}\lambda^n(1-G(0))^n\left(e^{-\lambda t-\int_0^t\beta(u)du}\right)^{*n}, -\lambda\leq\frac{\int_0^t\beta(u)du}{t}\leq\frac{\lambda}{e^{\rho}-1} \tag{4.5}$$

for the M|G|∞ busy cycle d. f..
If $\beta(t)=\beta$ (constant)

$$Z^{\beta}(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda-e^{-\rho}(\lambda+\beta)}e^{-e^{-\rho}(\lambda+\beta)t}-\frac{\beta}{\lambda-e^{-\rho}(\lambda+\beta)}e^{-\lambda t},$$

$$t\geq 0, -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \tag{4.6}$$

So, if the service time d. f. is given by (3.3) the M|G|∞ busy cycle d.f. is the mixture of two exponential distributions.

Finally note that, for $\beta=\frac{\lambda}{e^{\rho}-1}, Z^{\beta}(t)=1-\frac{\left(e^{\rho}-1\right)e^{-\frac{\lambda}{e^{\rho}-1}t}-e^{-\lambda t}}{e^{\rho}-2}, t\geq 0$.

And $Z(t)$, given by (4.5) satisfies

$$Z(t) \geq 1 - \frac{\left(e^{\rho}-1\right) e^{-\frac{\lambda}{e^{\rho}-1}t} - e^{-\lambda t}}{e^{\rho}-2}, t \geq 0, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1} \quad (4.7).$$

Note that (4.7) is coherent even for $\rho = \log 2$ since that

$$\lim_{\rho \to \log 2}\left(1 - \frac{\left(e^{\rho}-1\right) e^{-\frac{\lambda}{e^{\rho}-1}t} - e^{-\lambda t}}{e^{p}-2}\right) =$$

$$= \lim_{\rho \to \log 2} \frac{e^{\rho}-2-\left(e^{\rho}-1\right) e^{-\frac{\lambda}{e^{\rho}-1}t} + e^{-\lambda t}}{e^{\rho}-2} = 1-\left(1+\lambda t\right)e^{-\lambda t}.$$

Finally, we have also

$$Z(t) \leq 1-e^{-\lambda t}, t \geq 0, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1} \quad (4.8).$$

5 STUDY OF $\mu(1',t)$ AS TIME FUCTION

Proposition 5.1

If $G(t)<1, \ t>0$, continuous, differential and

$$h(t) \leq \lambda, \quad t>0 \qquad (5.1)$$

$\mu(1',t)$ is not increasing.

Dem:

It is enough to see, having account (1.3), that $\frac{d}{dt}\mu(1',t) = (1-G(t))(\lambda - h(t))$.

Notes:

If the rate at which the services finish is lower or equal to the arrival rate of customers $\mu(1',t)$ is not increasing;

For the M|M|∞ system (5.1) is equal to

$$\rho \geq 1 \quad (5.2).$$
$$\lim_{t\to\infty} \mu(1',t) = \rho.$$

Doing $h(t) - \lambda = \beta(t)$, any $\beta(.)$, we obtain, solving this first order linear differential equation,

$$G(t) = 1-(1-G(0))e^{-\lambda t - \int_0^t \beta(u)du}, t \geq 0, \frac{\int_0^t \beta(u)du}{t} \geq -\lambda \quad (5.3).$$

So we have the

Proposition 5.2

If $\beta = 0$
$G(t) = 1 - (1 - G(0))e^{-\lambda t}$, $t \geq 0$ (5.4)

and $\mu(1',t) = 1 - G(0) = \rho, \quad t \geq 0.$

Let's see now expressions of $\mu(1',t)$ to some particular distributions of services:

Deterministic of value α

$$\mu(1',t) = \begin{cases} 1 + \lambda t, t < \alpha \\ \rho, t \geq \alpha \end{cases} \qquad (5.5),$$

Exponential

$$\mu(1',t) = \rho + (1 - \rho)e^{-\frac{t}{\alpha}} \qquad (5.6),$$

Collection given by (3.3)

$$\mu(1't) = \frac{(1 - e^{-\rho})(\lambda + \beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t} - 1\right) + \lambda} + \rho - \log\left(1 + \left(e^{\rho} - 1\right)e^{-(\lambda+\beta)t}\right) \qquad (5.7).$$

6 STUDY OF $V(1',t)$ AS TIME FUNCTION

Proposition 6.1

If $G(t) < 1,\ t > 0,$ continuous, differential and

$$h(t) \geq -\frac{\lambda}{1 - 2G(t)} \qquad (6.1)$$

$V(1',t)$ is not decreasing.

Dem:
It is enough to see, having account (1.4), that

$$\frac{d}{dt}V(1',t) = \lambda(1 - G(t)) + g(t) - 2G(t)g(t) = \lambda(1 - G(t)) + g(t)(1 - 2G(t)) =$$
$$= (1 - G(t))(h(t)(1 - 2G(t)) + \lambda).$$

Notes:

Obviously $1 - 2G(t) < 0 \Leftrightarrow G(t) > \frac{1}{2}, t > 0.$

$$\lim_{t\to\infty} V(1',t) = \rho.$$

Doing $h(t)+\frac{\lambda}{1-2G(t)}=0$ we obtained, solving this first order differential equation, the

Proposition 6.2

If $G(.)$ is implicitly defined as

$$\frac{1-G(t)}{1-G(0)}e^{2(G(t)-G(0))}=e^{-\lambda t},t\geq 0 \qquad (6.2)$$

$$V(1',t)=\rho,t\geq 0.$$

Notes:

The density associated to (6.2) is given by

$$g(t)=-\frac{\lambda e^{-\lambda t}(1-G(0))}{(1-2G(t))e^{2(G(t)-G(0))}} \qquad (6.3)$$

Starting from (6.3), designating by S the random variable which is associated to it, we show easily that, with $G(0)>\frac{1}{2}$,

$$\frac{(1-G(0))n!e^{-2(1-G(0))}}{\lambda^n}\leq \mathrm{E}[S^n]\leq\frac{(1-G(0))n!}{(2G(0)-1)\lambda^n},n=1,2,... \qquad (6.4).$$

Let's see now expressions of $V(1',t)$ to some particular distributions of services:

Deterministic of value α

$$V(1',t)=\begin{cases}\lambda t,t<\alpha\\ \rho,t\geq\alpha\end{cases} \qquad (6.5),$$

Exponential

$$V(1',t)=\rho\left(1-e^{-t/\alpha}\right)+e^{-\frac{t}{\lambda}}+e^{-\frac{2t}{\alpha}} \qquad (6.6),$$

Collection given by (3.3)

$$V(1',t)=\rho-\log\left(1+\left(e^{\rho}-1\right)e^{-(\lambda+\beta)t}\right)+\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda}+\left(\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda}\right)^2 \qquad (6.7)$$

7 Concluding remarks

In queues practical applications often it is used the populational process stationary distribution. This happens generally because the transient distribution is very complex and unuseful. And so the stationary distribution is used as a good transient one approximation. But in various situations this is not true. So it is necessary to know as well as possible the transient behaviour.

The $M|G|\infty$ systems transient behaviour, with an unoccupied system instant time origin, is very well known and not too complex. We deduced the time origin at the beginning of a busy period transient distribution.

We presented here a transient behaviour study, with some interesting results, for the M|G|∞ systems with a lot of possible applications, namely in survival analysis.
It was done more exhaustively for the $p_{0n}(t)$ than for the $p_{1'n}(t), n = 0,1,2,...$, but in the former situation everything is easier than in the other. But the $p_{1'0}(t)$ study leads to very interesting results even that they are looked only from the mathematical point of view. And, no less important, it allows through the resolution of a Ricatti equation the determination of a M|G|∞ infinite systems collection with a very simple busy period distribution: a mixture of a degenerate distribution at the origin and an exponential distribution. And also for the M|G|∞ busy cycle.
It was also possible to study the behaviour of $\mu(1',t)$ and $V(1',t)$, as time functions, playing here the service hazard rate function a very important role. It allowed the consideration of linear differential equations that lead to very interesting results.

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