The Coefficient of Variation Asymptotic Distribution in Case of Non-IID Random Variables

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Abstract

Due to the coefficient of variation's widespread use in empirical finance, in this paper we derive its asymptotic sampling distribution in case of non-iid random variables to deal with autocorrelation and/or conditional heteroscedascity stylized facts of financial returns. We propose also statistical tests for the comparison of two coefficients of variation based on asymptotic normality and studentized time series bootstrap. In an illustrative example we analyze the monthly returns' volatility of six stock markets indexes during the years 1990-2007.

Keywords: Coefficient of variation, Autocorrelation, Conditional heteroscedasticity, Non-iid random variables.

JEL: C1, C5

1 Introduction

The variance and the standard deviation are the most commonly used dispersion measures in statistics and other application fields. When compared to the variance, the standard deviation has an important advantage: it is expressed in the same units of the variable under study, while the variance is measured in the square units of the respective variable. Thus, the standard deviation is easier to interpret.

However, when the main purpose is to compare the dispersion of several variables' distributions, the standard deviation is not the most appropriate indicator unless all the variables are expressed in the same measurement units and have identical mean values. When these two requirements do not hold, the coefficient of variation (CV) is the relative dispersion measure frequently used and it expresses the standard deviation as a proportion of the arithmetic mean:

$$CV = \frac{\sigma}{\mu} \, , \tag{1}$$

where μ and σ are the population mean and standard deviation of the variable distribution whose dispersion is under scrutiny and the result is often reported as a percentage (see, for example, Ahmed, 1994). The variable with the largest value for the coefficient of variation is the one with the highest relative dispersion around the mean. Note that the ratio makes no sense if the expected value is zero or negative. Thus, the coefficient of variation is useful for comparing the relative variability of strictly positive random variables distributions.

When the distribution is unknown, the parameters μ and σ can be estimated based on sample values and the estimator for the coefficient of variation is:

$$\widehat{\mathrm{CV}} = \frac{\hat{\sigma}}{\hat{\mu}}, \tag{2}$$

where $\hat{\sigma}$ and $\hat{\mu}$ are the sample estimates of the standard deviation and the arithmetic mean, respectively.

As the coefficient of variation is a unit-free measure of dispersion, it has been widely used rather than the standard deviation in many scientific areas (see Nairy and Rao (2003) for a brief survey of recent applications in business, engineering, climatology and other fields). Despite its widespread use, in this paper we concentrate mainly on the finance field coefficient of variation applications.

In finance the term "volatility" stands for risk and uncertainty and it is usually measured by the standard deviation (or a similar measure of dispersion) of the observed (or expected) prices and returns of financial assets. The greater is the variation in prices or returns the higher is the standard deviation which in turn is linked to higher risk. As the CV is more appropriate when the objective is to compare prices and returns volatility (risk) of alternative investments it has also been applied in several studies and a lower CV ratio represents a lower risk. In the next paragraph we present a few examples regarding the CV financial field applications.

Brief and Owen (1969) show how the CV can be considered in order to evaluate the projects risks, assuming the rate of return as a random variable. The authors used the CV of the future cash flows distribution as a measure of earnings risk and developed a mechanism for relating the CV to risk in a situation of uncertainty. Weinraub and Kuhlman (1994) tested the relationship between the variability of individual stock betas and the variability of a small portfolio. In this study they have used two measures of beta variability: the standard deviation and the coefficient of variation. In short, the coefficient of variation revealed an inverse relationship between the level of beta and the relative variability. According to this result, they argue that betas less than 1 are poor predictors of future returns. Boyle and Rao (2001) intended to clarify the conditions which justify the mean-generalized coefficient of variation analysis on a utility-theoretic basis. In this work the authors argue that in contrast to the standard deviation, this measure emphasizes the intuitive notion of 'downside' risk. Worthington and Higgs (2003) have used the CV to measure the degree of risk in relation to the mean return to study the portfolio diversification among major painting and financial markets over the period 1976-2001.

In theoretical terms, recent investigation related with the coefficient of variation (or its inverse) is driven by two main issues: the sampling distribution of $\widehat{\mathrm{CV}}$ and hypotheses testing for coefficients of variation comparison (Argaç, 2005).

First, the finite sample and the asymptotic sampling distributions of $\widehat{\mathrm{CV}}$ are needed for the statistical inference about the population CV. Under iid and normality assumptions, Iglewicz (1967) derived the expected value and the variance of $\widehat{\mathrm{CV}}$ as well as the exact distribution of the sample coefficient of variation: $\sqrt{T}\frac{1}{\widehat{\mathrm{CV}}}$ has a non-central Student's t distribution with $T-1$ degrees of freedom and non-centrality parameter $\sqrt{T}\frac{1}{\mathrm{CV}}$, where T is the number of observations. Sharma and Krishna (1994) developed (under iid assumption) the asymptotic distribution of the inverse of the coefficient of variation discarding the normality assumption of the population distribution. They show that $1/\widehat{CV}$ is an asymptotically unbiased and $s-$consistent estimator of $1/CV$ and $\sqrt{T}\left(1/\widehat{CV}-1/CV\right)$ is asymptotically standard s-normal in distribution.

Second, when the objective is to compare the distributions dispersion around the mean, as the observed differences in the estimated CVs resulting from different samples can be due to sampling error, it becomes necessary to test if that differences are statistically significant and several tests have been proposed in the literature to compare the CVs of k normal populations. Nairy and Rao (2003) divide these tests in three main categories: likelihood ratio tests (Miller and Karson, 1977; Bennett, 1977; Doornbos and Dijkstra, 1983; Gupta and Ma, 1996), Wald tests (Rao and Vidya, 1992; Gupta and Ma, 1996) and score tests (Gupta and Ma, 1996). Other relevant contributions for this subject are due to Miller (1991) and Miller and Feltz (1997).

In all these statistical tests it is assumed that random variables $X_1, X_2, \ldots, X_T$ are independently and identically distributed. However, when the application field is finance, the dependence of high frequency data (daily or monthly returns, for example) under different ways: autocorrelation and/or conditional heteroscedasticity, becomes the traditional inference approach inappropriate.

Due to this financial data characteristics, in this paper we derive the asymptotic sampling distribution of the coefficient of variation in case of non-iid random variables and we derive statistical tests for the comparison of two coefficients of variation. In the next two sections we provide the procedure's description, then we make and empirical application and the final section summarizes our concluding remarks.

2 The asymptotic distribution of the CV estimator

In this section we derive explicit expressions for the statistical distribution of the coefficient of variation using standard asymptotic theory under iid and non-iid assumptions. This distribution completely characterizes the statistical behavior of $\widehat{\mathrm{CV}}$ in large samples and allows us to quantify the $\widehat{\mathrm{CV}}$ precision to estimate CV. First we discuss the asymptotic distribution under the standard assumption that $X_1, X_2, \ldots, X_T$ are iid random variables. Next, as the iid condition is extremely restrictive and empirically implausible in financial data (see, for example, Curto et al., 2007) a more general distribution is derived under the non-iid assumption.

2.1 IID assumption

If $X_1, X_2, \ldots, X_T$ are iid random variables with finite mean μ and variance σ^2 the estimators $\hat{\mu}$ and $\hat{\sigma}^2$ have the following normal distribution in large samples due to the Central Limit Theorem (White, 2001):

$$\sqrt{T}\,(\hat{\mu}-\mu) \overset{a}{\sim} N(0,\sigma^2), \quad \sqrt{T}\left(\hat{\sigma}^2-\sigma^2\right) \overset{a}{\sim} N(0,2\sigma^4), \tag{3}$$

where $\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} X_t$, $\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}(X_t-\hat{\mu})$ and $\overset{a}{\sim}$ denotes asymptotically, i.e., as T increases without bound, the probability distributions of $\sqrt{T}\,(\hat{\mu}-\mu)$ and $\sqrt{T}\left(\hat{\sigma}^2-\sigma^2\right)$ approach the normal distribution.

To derive the asymptotic distribution of $\widehat{\mathrm{CV}}$ we follow Lo (2002)[1] and the first step is to obtain the asymptotic joint distribution of $\hat{\mu}$ and $\hat{\sigma}^2$. Denote by $\hat{\theta}$ the column vector $\begin{pmatrix}\hat{\mu} & \hat{\sigma}^2\end{pmatrix}'$ and by θ the corresponding column vector of population values $\begin{pmatrix}\mu & \sigma^2\end{pmatrix}'$. An additional property of $\hat{\mu}$ and $\hat{\sigma}$ under the iid assumption is that they are statistically independent in large samples. Thus,

$$\sqrt{T}\left(\hat{\theta}-\theta\right) \overset{a}{\sim} N\left(0, V_\theta\right), \quad V_\theta \equiv \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}. \tag{4}$$

Because the coefficient of variation estimator $\widehat{\mathrm{CV}}$ can be written as a function $f(\hat{\theta})$ of $\hat{\theta}$ it can be directly applied the delta method (White, 2001) to derive its asymptotic distribution.

If $\sqrt{T}\left(\hat{\theta}-\theta\right) \overset{a}{\sim} N\left(0, V_\theta\right)$, then a nonlinear function $f(\hat{\theta})$ has the following asymptotic

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[1] Lo derives several statistics of Sharpe ratio, one of the most important financial measures of portfolio performance based on risk-adjusted excess returns. The Sharpe ratio is defined as the expected excess return divided by the return standard deviation. Thus, it is the inverse of the Coefficient of Variation in the particular case of financial returns.

distribution:

$$\sqrt{T}\left[f(\hat{\theta})-f(\theta)\right] \overset{a}{\sim} N(0,V_f), \quad \text{where} \quad V_f \equiv \frac{\partial f(\theta)}{\partial \theta} V_\theta \frac{\partial f(\theta)}{\partial \theta'}. \tag{5}$$

In case of the coefficient of variation,

$$V_\theta \equiv \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}, \quad f(\theta) = \frac{\sigma}{\mu}, \quad \frac{\partial f(\theta)}{\partial \theta'} = \begin{bmatrix} -\sigma/\mu^2 \\ 1/2\sigma\mu \end{bmatrix} \tag{6}$$

and it follows that the asymptotic distribution of $\widehat{\text{CV}}$ is:

$$\sqrt{T}\left(\widehat{\text{CV}} - CV\right) \overset{a}{\sim} N(0, V_{IID}), \qquad V_{IID} = CV^4 + \frac{1}{2}CV^2. \tag{7}$$

where the asymptotic variance is given by the weighted average of the asymptotic variances of $\hat{\mu}$ and $\hat{\sigma}^2$:

$$V_{IID} = \left(\frac{\partial f}{\partial \mu}\right)^2 \sigma^2 + \left(\frac{\partial f}{\partial \sigma^2}\right)^2 2\sigma^4 = \left(-\frac{\sigma}{\mu^2}\right)^2 \sigma^2 + \left(\frac{1}{2\sigma\mu}\right)^2 2\sigma^4 = CV^4 + \frac{1}{2}CV^2. \tag{8}$$

The weights are the squared sensitivities of f with respect to μ and σ^2 and the more sensitive f is to a particular parameter, the more influential its asymptotic variance will be in the asymptotic variance of the coefficient of variation.

Therefore, standard errors for the coefficient of variation estimator $\widehat{\text{CV}}$ can be computed as:

$$\text{SE}(\widehat{\text{CV}}) \overset{a}{=} \sqrt{\left(CV^4 + \frac{1}{2}CV^2\right)/T}, \tag{9}$$

and this quantity can be estimated by replacing $\widehat{\text{CV}}$ for CV. For any given sample size T, larger coefficients of variation imply larger standard errors. Confidence intervals can also be constructed for CV around the estimator $\widehat{\text{CV}}$:

$$\widehat{\mathrm{CV}} \pm z_{(1-\frac{\alpha}{2})}\sqrt{\left(CV^4 + \frac{1}{2}CV^2\right)/T}, \tag{10}$$

where $z_{(1-\frac{\alpha}{2})}$ is the $(1-\frac{\alpha}{2})$ quantile of the standard normal distribution.

2.2 Non-IID assumption

When the $X_1, X_2, \ldots, X_T$ iid assumption does not hold, the results of the previous section may be of limited practical value and the asymptotic distribution can be derived by using a "robust" estimator for the coefficient of variation.

Following Lo (2002) we apply the generalized method of moments (GMM) to estimate μ and σ^2 and the results of Hansen (1982) can be used to derive the asymptotic distribution of the coefficient of variation. Hansen shows that:

$$\sqrt{T}\left(\widehat{\theta} - \theta\right) \overset{a}{\sim} N\left(0, V_\theta\right), \quad \text{where} \;\; V_\theta \equiv H^{-1}\Sigma\left(H^{-1}\right)', \tag{11}$$

$$H \equiv \lim_{T\to\infty} E\left[\frac{1}{T}\sum_{t=1}^{T}\varphi_\theta\left(X_t, \theta\right)\right], \quad \Sigma \equiv \lim_{T\to\infty} E\left[\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\varphi\left(X_t, \theta\right)\varphi\left(X_s, \theta\right)'\right], \tag{12}$$

and $\varphi_\theta\left(X_t, \theta\right)$ represents de derivative of $\varphi\left(X_t, \theta\right)$ with respect to θ. Let $\varphi\left(X_t, \theta\right)$ denotes the vector function with the following moment conditions:

$$\varphi\left(X_t, \theta\right) = \begin{bmatrix} X_t - \mu \\ \left(X_t - \mu\right)^2 - \sigma^2 \end{bmatrix}. \tag{13}$$

The GMM estimator of θ is given by the solution to:

$$\frac{1}{T}\sum_{i=1}^{T}\varphi\left(X_t, \theta\right) = 0, \tag{14}$$

yielding the standard estimators $\hat{\mu}$ and $\hat{\sigma}^2$ defined before.

For the moments conditions in (13), the corresponding matrix with the derivatives is:

$$H \equiv \lim_{T\to\infty} E\left\{\frac{1}{T}\sum_{t=1}^{T}\left[\begin{array}{cc} -1 & 0 \\ 2\left(\mu - X_t\right) & -1 \end{array}\right]\right\} = -I. \tag{15}$$

Therefore, $V_\theta \equiv \Sigma$ and the asymptotic distribution of the coefficient of variation estimator follows from the delta method as shown in the previous section:

$$\sqrt{T}\left(\widehat{\mathrm{CV}} - CV\right) \overset{a}{\sim} N\left(0, V_{GMM}\right), \quad \text{where} \quad V_{GMM} = \frac{\partial f\left(\theta\right)}{\partial \theta}\Sigma\frac{\partial f\left(\theta\right)}{\partial \theta'}. \tag{16}$$

In order to estimate the asymptotic variance, an estimator for $\frac{\partial f(\theta)}{\partial \theta}$ may be obtained by substituting $\hat{\theta}$ into equation (6) and an heteroscedasticity and autocorrelation[2] consistent (HAC) estimator $\hat{\Sigma}$ may be obtained by using the Newey and West's (1987) procedure:

$$\hat{\Sigma} = \hat{\Omega}_0 + \sum_{j=1}^{m}\omega(j,m)\left(\hat{\Omega}_j + \hat{\Omega}_j'\right), m \ll T, \tag{17}$$

$$\hat{\Omega}_j \equiv \frac{1}{T}\sum_{t=j+1}^{T}\varphi\left(X_t, \hat{\theta}\right)\varphi\left(X_t, \hat{\theta}\right)',$$

$$\omega(j,m) = 1 - \frac{j}{m+1},$$

where m is the truncated lag that must satisfy the condition $m/T \to \infty$ as T increases without bound to ensure consistency.

Therefore, for non-iid random variables, the standard error of the coefficient of variation can be estimated by $SE\left(\widehat{CV}\right) \overset{a}{=} \sqrt{\frac{V_{GMM}}{T}}$ and confidence intervals for CV can be constructed in a similar fashion to equation (10):

$$\widehat{\mathrm{CV}} \pm z_{(1-\frac{\alpha}{2})} SE\left(\widehat{CV}\right).$$

[2]Heteroscedasticity and/or autocorrelation of unknown form are often important specification issues, specially in macroeconomics and financial applications.

3 Tests for the coefficients of variation comparison

As we referred before, the hypotheses testing for coefficients of variation comparison is also an important statistical issue. In order to test the difference between two coefficients of variation, consider the bidimensional variable (X_{1t}, X_{2t}), for $t = 1, 2, \ldots, T$, whose distribution has mean vector μ and covariance matrix Ψ given by:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{and} \quad \Psi = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}. \tag{18}$$

The difference between the two coefficients of variation is given by

$$\Delta = CV_1 - CV_2 = \frac{\sigma_1}{\mu_1} - \frac{\sigma_2}{\mu_2} \tag{19}$$

and its estimator is

$$\widehat{\Delta} = \widehat{CV}_1 - \widehat{CV}_2 = \frac{\hat{\sigma}_1}{\hat{\mu}_1} - \frac{\hat{\sigma}_2}{\hat{\mu}_2}. \tag{20}$$

Furthermore, let $\theta = \left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2\right)'$, $\hat{\theta} = \left(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1^2, \hat{\sigma}_2^2\right)'$ and

$$\frac{\partial \Delta}{\partial \theta'} = \left(-\frac{\sigma_1}{\mu_1^2}, \frac{\sigma_2}{\mu_2^2}, \frac{1}{2\sigma_1\mu_1}, -\frac{1}{2\sigma_2\mu_2}\right)'. \tag{21}$$

3.1 IID assumption

If $X_{j1}, X_{j2}, \ldots, X_{jT},\ j = 1, 2$, are iid random variables and (X_{1t}, X_{2t}) has a bivariate normal distribution, Jobson and Korkie (1981) and Memmel (2003) shows that

$$\sqrt{T}\left(\hat{\theta} - \theta\right) \overset{a}{\sim} N\left(0, V_\theta\right), \quad V_\theta \equiv \begin{bmatrix} \sigma_1^2 & \sigma_{12} & 0 & 0 \\ \sigma_{12} & \sigma_2^2 & 0 & 0 \\ 0 & 0 & 2\sigma_1^4 & 2\sigma_{12}^2 \\ 0 & 0 & 2\sigma_{12}^2 & 2\sigma_2^4 \end{bmatrix}. \tag{22}$$

Applying the delta method,

$$\sqrt{T}\left(\widehat{\Delta}-\Delta\right)\overset{a}{\sim}N\left(0,V_{\Delta IID}\right), \tag{23}$$

where

$$V_{\Delta IID}=CV_1^4+CV_2^4+\frac{1}{2}CV_1^2+\frac{1}{2}CV_2^2-2\frac{\sigma_1\sigma_2\sigma_{12}}{\mu_1^2\mu_2^2}-\frac{\sigma_{12}^2}{\mu_1\mu_2\sigma_1\sigma_2}.$$

Replacing the parameters by their estimators it is also possible to construct confidence intervals for Δ:

$$\widehat{\Delta}\pm z_{(1-\frac{\alpha}{2})}\sqrt{\left(\widehat{CV}_1^4+\widehat{CV}_2^4+\frac{1}{2}\widehat{CV}_1^2+\frac{1}{2}\widehat{CV}_2^2-2\frac{\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_{12}}{\hat{\mu}_1^2\hat{\mu}_2^2}-\frac{\hat{\sigma}_{12}^2}{\hat{\mu}_1\hat{\mu}_2\hat{\sigma}_1\hat{\sigma}_2}\right)/T}, \tag{24}$$

where $z_{(1-\frac{\alpha}{2})}$ is the $(1-\frac{\alpha}{2})$ quantile of the standard normal distribution.

If zero is not contained in the resulting interval we conclude that the difference between the coefficients of variation is statistically significant.

3.2 Non-IID assumption

As the formula in (22) is no longer valid if the bivariate distribution is not normal or if the observations are correlated, in this section we derive the asymptotic distribution of $\widehat{\Delta}$ considering an heteroskedasticity and autocorrelation consistent (HAC) estimator for Σ and we apply the Wolf (2007) studentized bootstrap method to test the nullity of Δ.

Let $\varphi\left(X,\theta\right)$ denotes the vector function with the following moment conditions:

$$\varphi\left(X,\theta\right)=\begin{bmatrix} X_{1t}-\mu_1 \\ X_{2t}-\mu_2 \\ \left(X_{1t}-\mu_1\right)^2-\sigma_1^2 \\ \left(X_{2t}-\mu_2\right)^2-\sigma_2^2 \end{bmatrix}. \tag{25}$$

For the moments conditions in (25), the corresponding matrix with the derivatives is:

$$H \equiv \lim_{T\to\infty} E\left\{\frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2\left(\mu_1 - X_{1t}\right) & 0 & -1 & 0 \\ 0 & 2\left(\mu_2 - X_{2t}\right) & 0 & -1 \end{bmatrix}\right\} = -I. \tag{26}$$

Therefore, $V_\theta \equiv \Sigma$ and the asymptotic distribution of $\widehat{\Delta}$ follows from the delta method:

$$\sqrt{T}\left(\widehat{\Delta} - \Delta\right) \overset{a}{\sim} N\left(0, V_{\Delta GMM}\right), \quad \text{where } V_{\Delta GMM} = \frac{\partial \Delta}{\partial \theta}\Sigma\frac{\partial \Delta}{\partial \theta'}. \tag{27}$$

In order to estimate the asymptotic variance, an estimator for $\frac{\partial \Delta}{\partial \theta}$ may be obtained by substituting $\hat{\theta}$ into equation (21) and an heteroscedasticity and autocorrelation consistent (HAC) estimator $\hat{\Sigma}$ may be obtained by using the Newey and West's (1987) procedure.

Thus, the standard error $s\left(\widehat{\Delta}\right) \overset{a}{=} \sqrt{\frac{V_{\Delta GMM}}{T}}$ combined with the asymptotic normality in (27) allows the HAC inference as follows. A two-sided p-value for the null hypothesis $H_0 : \Delta = 0$ is given by

$$\hat{p} = 2\Phi\left[-\frac{|\widehat{\Delta}|}{s\left(\widehat{\Delta}\right)}\right]$$

where $\Phi(\cdot)$ denotes the c.d.f. of the standard normal distribution. Alternatively, it is possible to compute a $1-\alpha$ confidence interval for Δ:

$$\widehat{\Delta} \pm z_{(1-\frac{\alpha}{2})} \times s\left(\widehat{\Delta}\right).$$

However, when data is heavy-tailed (non-normal) or of time series nature, HAC inference is often liberal when samples sizes are small to moderate. This means that hypothesis tests tend to reject a true null hypothesis too often; see, for example, Andrews (1991), Andrews (1992) and Romano and Wolf (2006). Thus, included in the extensive literature that demonstrates the improved inference accuracy of the studentized bootstrap over standard inference based on asymptotic normality, Wolf (2007) propose to test the equality of two Sharpe ratios by inverting a bootstrap confidence interval. If this interval does not contain zero, then H_0 is rejected at the nominal significance level α.

In this paper we also apply the Wolf (2007) studentized time series bootstrap (BOOT-ST) method to test the coefficients of variation equality by constructing a symmetric studentized bootstrap confidence interval. Let the two-sided distribution function of the studentized statistic be approximated via the bootstrap as follows:

$$F\left[\frac{|\widehat{\Delta}-\Delta|}{s\left(\widehat{\Delta}\right)}\right] \approx F\left[\frac{|\widehat{\Delta}^{*}-\widehat{\Delta}|}{s\left(\widehat{\Delta}^{*}\right)}\right], \tag{28}$$

where Δ is the true difference between the coefficients of variation, $\widehat{\Delta}$ is the estimated difference computed from original data, $s\left(\widehat{\Delta}\right)$ is a standard error for $\widehat{\Delta}$ (also computed from the original data), $\widehat{\Delta}^{*}$ and $s\left(\widehat{\Delta}^{*}\right)$ are the estimated difference and the standard error computed from bootstrap data and $F(X)$ denotes the distribution function of the random variable X.

Letting $z^{*}_{|\cdot|,\lambda}$ be a λ quantile of $F\left(|\widehat{\Delta}^{*}-\Delta|/s\left(\widehat{\Delta}^{*}\right)\right)$, a bootstrap $1-\alpha$ confidence interval for Δ is then given by

$$\widehat{\Delta} \pm z^{*}_{|\cdot|,1-\alpha} \times s\left(\widehat{\Delta}\right). \tag{29}$$

In case of heavy-tailed or time series data, $z^{*}_{|\cdot|,1-\alpha}$ will be typically larger than $z_{(1-\frac{\alpha}{2})}$ for small to moderate samples, resulting in more conservative inference compared to the HAC methods.

Since the confidence interval is constructed, the hypothesis $H_0: \Delta = 0$ is rejected if the value zero is not in the interval. However, it might be more desirable to obtain a p-value and Wolf (2007) proposes a simple method to compute it. Let d denotes the studentized test statistic based on original data:

$$d = \frac{|\widehat{\Delta}|}{s\left(\widehat{\Delta}\right)}$$

and represent the *centered* studentized statistic computed from the mth bootstrap sample by $\tilde{d}^{*,m}, m = 1, 2, \ldots, M$

$$\tilde{d}^{*,m} = \frac{|\widehat{\Delta}^{*,m} - \widehat{\Delta}|}{s\left(\widehat{\Delta}^{*,m}\right)},$$

where M is the number of bootstrap resamples. Then the p-value is given by:

$$\hat{p} = \frac{\#\left\{\tilde{d}^{*,m} \geq d\right\} + 1}{M + 1}. \tag{30}$$

4 Empirical application

4.1 Statistical properties of returns

The data consists of monthly closing prices of the S&P, DJIA, NASDAQ, CAC40, DAX30 and FTSE100 (source: Yahoo! Finance), which are main indexes for the US (S&P500, DJIA, and NASDAQ), France, German and UK equity markets, respectively. These series cover the period from December 1, 1989 to December 31, 2007 yielding 205 monthly observations. We analyze the continuously compounded percentage rates of return (adjusted for dividends) that are calculated by taking the first differences of the logarithm of series (P_t is the closing value for each stock index at month t):

$$r_t = 100 \times \left[\ln\left(P_t\right) - \ln\left(P_{t-1}\right)\right]. \tag{31}$$

Due to this differencing process, the number of observations reduces to 204. Table 1 summarizes the basic statistical properties of the data. The mean returns are all positive but less than 1%. The monthly returns appear to be somewhat asymmetric and leptokurtic as reflected by negative skewness and excess of kurtosis estimates (skewness and kurtosis coefficients are all statistically different from those of the standard Normal distribution which are 0 and 3, respectively). The Jarque-Bera test also rejects the null hypothesis of normality. It is also interesting to observe that different conclusions about the returns' dispersion would be taken if different measures are considered. In fact, using the standard deviation, the FTSE100 seems to present the smaller variability. However, when the coefficient of variation is considered, the smaller variability is associated to DJIA. It is also interesting to notice that apparently the standard deviation points to a smaller returns' variability when compared to the coefficient of variation. Obviously, this happens because all the mean returns are less than 1%, an empirical result that is common in monthly data. As CV is the ratio between the standard deviation and the mean, it reflects better the returns' variability as we have explained before.

According to the Ljung-Box statistic for returns, there is no relevant autocorrelation for all the stock indexes. Even though the series of returns seems to be serially uncorrelated over time, the Ljung-Box statistic for up to twelve order serial correlation of squared returns is highly significant at any level for the six stock indexes, suggesting the presence of strong nonlinear dependence in the data. As non-linear dependence and heavy-tailed unconditional distributions are characteristic of conditionally heteroskedastic data, the Lagrange Multiplier test (Engle, 1982) can be used to formally test the presence of conditional heteroscedasticity. The LM test for a twelve-order (in the last row of table 1) suggests that all stock indexes' returns exhibit conditional heteroscedasticity, implying that nonlinearities must enter through the variance of

Table 1: Summary statistics of returns

Statistics	S&P500	DJIA	NASDAQ	CAC40	DAX30	FTSE100
Mean	0.731	0.793	0.961	0.644	0.859	0.541
Median	1.115	1.107	1.740	1.312	1.496	0.850
Maximum	10.579	10.079	19.865	12.588	19.374	9.890
Minimum	-15.759	-16.407	-26.009	-19.225	-29.333	-12.736
Std. Dev.	3.875	3.949	6.947	5.367	6.112	3.871
Coef. of Var.	5.297	4.983	7.232	8.333	7.114	7.162
Skewness	-0.659	-0.686	-0.761	-0.521	-0.918	-0.587
Kurtosis	4.441	4.899	5.086	3.746	6.374	3.835
Jarque-Bera	32.437	46.647	56.670	13.947	125.454	17.625
Probability	0.000	0.000	0.000	0.001	0.000	0.000
LB Q(12)[a]	9.379	9.351	13.05	13.035	12.665	6.026
Probability	0.670	0.673	0.365	0.367	0.394	0.915
LB $Q^2(12)$[b]	32.767	25.695	179.040	40.736	44.642	31.496
Probability	0.001	0.012	0.000	0.000	0.000	0.002
ARCH(12) LM[c]	22.229	21.730	51.424	35.115	30.167	26.995
Probability	0.035	0.041	0.000	0.000	0.003	0.008

[a]LB Q(10) is the Ljung-Box test for returns,
[b]LB $Q^2(10)$ is the Ljung-Box test for squared returns,
[c]LM is the Engle's Lagrange Multiplier test for heteroscedasticity.

the processes.

As there is no evidence of autocorrelation in returns, we also use the White (1980) heteroscedasticity consistent (HC) estimator in (16) to derive the asymptotic variance of $\widehat{CV}$ and thus, $\hat{\Sigma} = \hat{\Omega}_0$.

4.2 Robust standard errors for the CV estimator

Table 2 shows the estimates for V_{IID}, V_{GMM}, $\widehat{CV}$ standard errors under iid and non-iid assumptions as well as confidence intervals for the monthly returns coefficients of variation for each one of the six stock indexes[3]. As one can see, the magnitudes of the standard errors under iid assumption yield 95% confidence intervals for coefficients of variation that do not contain 0 for five stock indexes (the exception is CAC40). These results indicate coefficients of variation for the monthly returns that are statistically different from 0 at the 95% confidence level. However, when the heteroscedasticity robust standard errors are considered, the CV estimates for NASDAQ, DAX30 and FTSE100 become statistically insignificant. These are the stock indexes where the conditional heteroscedasticity is more pronounced as we can see in table 1. Thus, when the heteroscedasticity is not accounted for, the iid standard errors leads to misleading conclusions in terms of the significance of the CVs estimates. After heteroscedasticity correction, just two of the coefficients of variation (SP&500 and DJIA) remain statistically significant. These are also the stock indexes with smaller CV estimates and higher confidence intervals precision.

Table 2: Robust standard errors and confidence intervals

Statistics	S&P500	DJIA	NASDAQ	CAC40	DAX30	FTSE100
Mean	0.731	0.793	0.961	0.644	0.859	0.541
Variance	15.013	15.594	48.256	28.804	37.353	14.988
Coefficient of variation	5.297	4.983	7.232	8.333	7.114	7.162
V_{IID}	801.435	628.844	2761.931	4857.171	2586.261	2656.765
V_{GMMHC}	909.5571	725.5039	3077.079	5171.362	2959.522	2882.979
V_{GMMHAC} $(m = 6)$	940.939	599.439	3840.324	6018.830	3823.106	3012.294
Confidence intervals IID						
Lower	1.412	1.542	0.020	-1.230	0.135	0.089
Upper	9.182	8.424	14.444	17.897	14.092	14.235
Confidence intervals HC						
Lower	1.159	1.287	-0.380	-1.535	-0.351	-0.206
Upper	9.436	8.679	14.844	18.201	14.579	14.530
Confidence intervals HAC						
Lower	1.088	1.623	-1.272	-2.313	-1.371	-0.369
Upper	9.507	8.343	15.736	18.979	15.599	14.694

[3] All the results are obtained using EVIEWS-based custom software.

4.3 Pairwise comparison tests

We performed next the CV pairwise comparison tests between the S&P500 (SP) and the remaining five stock indexes. The p-values associated with the test values are presented in table 3.

Table 3: p-values for various methods ($H_0 : \Delta = 0$)

Methods	SP-DJIA	SP-NASDAQ	SP-CAC40	SP-DAX30	SP-FTSE100
IID	0.678	0.419	0.428	0.484	0.462
GMMHC	0.691	0.432	0.435	0.494	0.466
GMMHAC	0.739	0.492	0.444	0.536	0.474
BOOT-TS	0.858	0.534	0.462	0.571	0.501

The equality assumption is always accepted despite the standard errors (under iid and non-iid assumptions, including Boot-TS) that we are considering (similar results were obtained when comparisons involved other indexes). These results can be explained by the extreme values of the asymptotic variances for the CV estimators that make the observed differences in the CVs estimates statistically insignificant. The high returns' variances must be due to the strong variability that characterizes the equity markets, special in "bear" times, and similar results are expected in terms of returns' volatility (measured by the CVs) due to the high integration of world equity markets. So, these results are not surprising. As the BOOT-TS p-values are higher when compared to those resulting from HAC methods, we confirm that in case of heavy-tailed data the inference based on BOOT-TS is more conservative.

5 Concluding remarks

The coefficient of variation is the ratio of standard deviation to the arithmetic mean and provides an important and widely used unit-free measure of dispersion, which can be used in comparing two distributions of different types with respect to their variability. As the iid assumption is extremely restrictive and often violated by financial data, in this paper we derive a more general distribution in case of non-iid random variables and we propose statistical tests for two coefficients of variation comparison.

In an illustrative example we compare the monthly returns volatility of six stock markets indexes during the years 1990-2007 and we show that when the heteroscedasticity is not accounting for, this can leads to misleading results in terms of the significance of the CVs estimates.

From the six stock indexes that we have considered, the CVs estimates are statistically significant in cases of SP&500 and DJIA. These are the stock indexes with the smallest values for the CVs estimates but they are also the stock indexes with the smallest asymptotic variances leading to more precise confidence intervals.

On the other hand, the CVs estimates (and the corresponding volatility) is higher in NASDAQ, CAC40, DAX30 and FTSE100 stock indexes but, due to the wider confidence intervals, the estimated values are not statistically significant.

When we compare the CVs estimated values of different stock indexes, the null equality assumption is always accepted and this result can be theoretically explained by the strong integration of world equity markets leading to similar results in terms of returns' volatility.

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