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METRIC FUNCTIONALS FOR THE HÄSTÖ METRIC

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ABSTRACT. In 2002, new classes of weighted metrics on $\mathbb{R}^n$ were introduced by Peter Hästö. In this article we compute the metric functionals for such classes of metrics.

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1. Introduction

Given a noncompact topological space there are several ways to compactify it. Among them, the one-point compactification is arguably the simplest. Another compactification technique, when the space is metrizable, is the metric compactification (see the definition below). A simple example of a metric compactification of the real line is given in example 15 of [3]. In this case, metric compactification amounts to the one-point compactification of $\mathbb{R}$. We observe that even this simple example has deep implications (see the unpublished preprint by Anders Karlsson and Nicolas Monod, available on Karlsson's web page). Thus, we believe that it is important to have concrete examples at hand where the metric compacification is known. Our result, despite being elementary, is a rather more elaborate case where the metric compactification of $\mathbb{R}^n$, for classes of weighted metrics studied by Hästö (which we call Hästö metrics) is also the one-point compactification.

Let (X, d) be a metric space with an arbitrary base point $x_o \in X$. Define a map $\Phi : X \to \mathbb{R}^X$ as follows

$$x \mapsto h_x(\cdot) = d(\cdot, x) - d(x_o, x). \tag{1.1}$$

The map Φ is injective and, if one considers the pointwise convergence on $\mathbb{R}^X$, it is also continuous. Identify X with the image $\Phi(X)$ and set

$$\overline{X} := \overline{\Phi(X)} = \overline{\{h_x : x \in X\}},$$

the pointwise closure. Following [3], $\overline{X}$ is called the metric compactification of (X, d) and the elements h of $\overline{X}$ are called metric functionals. Elements of the form (1.1) are called internal. The non internal metric functionals, that is, elements from $\overline{X}\backslash X$ form the boundary of the metric compactification.

Metric compactification can be traced back to Gromov who used instead uniform convergence on bounded sets. In this context, the common terminology for metric functionals is horofunctions. When the space X is proper both notions coincide, see [3] page 5. There are several papers regarding metric and horofunctions compactification of metric spaces and groups endowed with a metric. We will name just a few and suggest the interested reader to look into their bibliography for more references. For instance, in [6], the author computed the horoboundary of finite dimensional normed spaces. In a different setting, Ledrappier and Lim studied the horofunctions compactification of the Heisenberg group with the Korányi metric, see the unpublished preprint [5] (see also [4]). More recently, Gutiérrez [1] characterizes the metric compactification of L_p spaces by random measures. The recent work of Karlsson [3] is an excellent reference about the new ideas of an entire programme that seeks to apply techniques of functional analysis in metric spaces which fail to be linear. In particular, metric functionals as defined above are the analogue of linear functionals.

In this paper we completely characterize the metric compactification of the metric spaces $(\mathbb{R}^n, d_q)$ and $(\mathbb{R}^n, d_{p,q})$, where d_q and $d_{p,q}$ are the one parameter and two parameters Hästö metrics, two classes of weighted metrics introduced in [2]. We remark that none of these metrics comes from a norm, so the results of Walsh [6] do not apply. Specifically, we show that when $q \in]0, 1[$, the boundary of the metric compactification of $(\mathbb{R}^n, d_q)$ is $\{0\}$, see Theorem 2.1. Depending on parameters p and q, the boundary of the metric compactification of $(\mathbb{R}^n, d_{p,q})$ is always a singleton not necessarily zero, see Theorem 2.2.

2. Metric functionals for Hästö Metrics

We begin with the first class of Hästö metric over $\mathbb{R}^n$.

2.1. **One parameter Hästö metrics.** We use the terminology "one parameter" Hästö metric to refer to the class of metrics

$$d_q(x,y) = \frac{|x-y|}{(\max\{|x|,|y|\})^q}$$

where $x, y \in \mathbb{R}^n$ and $q \in]0,1]$, given in [2]. The value $d_q(0,0)$ is defined to be 0. We can state our first result:

Theorem 2.1. *Consider the metric space* $(\mathbb{R}^n, d_q)$. *When* $q \in]0,1[$ *the boundary of the metric compactification is* $\{0\}$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^n$. We start by noting that if $x_n \neq 0$ we have

$$d_q(x_n,0) = \frac{|x_n|}{(\max\{|x_n|,0\})^q} = |x_n|^{1-q}.$$

If $q = 1$ this quotient is just 1, but if $q \in]0,1[$ we have that $d_q(x_n,0)$ diverge to $+\infty$ if, and only if $|x_n|$ diverge to $+\infty$. We have

$$h_{x_n}(y) = d_q(x_n,y) - d_q(x_n,0) = \frac{|x_n-y|}{(\max\{|x_n|,|y|\})^q} - |x_n|^{1-q}.$$

Since we are interested in sequences $(x_n)_n$ such that $d(x_n,0)$ diverge to $+\infty$ we have that, for some order, $\max\{|x_n|,|y|\} = |x_n|$. Accordingly,

$$\begin{aligned} h_{x_n}(y) &= \frac{1}{|x_n|^q}(|x_n-y| - |x_n|) \\ &= |x_n|^{1-q}\left(\left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right| - 1\right). \end{aligned}$$

From the inequality $|a+b| \leq |a| + |b|$, defining c as $a+b$ we have $|c| \leq |a| + |c-a|$, or, $|c| - |a| \leq |c-a|$. Hence

$$1 - \left|\frac{y}{x_n}\right| \leq \left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right| \leq 1 + \left|\frac{y}{x_n}\right|$$

and

$$-\left|\frac{y}{x_n}\right| \leq \left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right| - 1 \leq \left|\frac{y}{x_n}\right|.$$

We are now able to compute the metric functional. From

$$-\frac{|y|}{|x_n|^q} \leq h_{x_n}(y) \leq \frac{|y|}{|x_n|^q},$$

and as $|x_n|$ diverge to infinity,

$$\lim_{x_n \to +\infty} h_{x_n}(y) = 0$$

∎

We now consider another class of Hästö metric.

2.2. **Two parameter Hästö metrics.** We use the terminology "two parameter" Hästö metrics to refer to the class of metrics

$$d_{p,q}(x,y) = \frac{|x-y|}{\left(\sqrt[p]{|x|^p + |y|^p}\right)^q}$$

where $x, y \in \mathbb{R}^n$, $q \in]0,1]$ and $p \geq \max\{1-q, (2-q)/3\}$, given in [2]. The value $d(0,0)$ is defined to be 0. In this setting our result is the following:

Theorem 2.2. *Consider the metric space* $(\mathbb{R}^n, d_{p,q})$. *We have*

(1) *when* $q \in]0, 1/2]$ *and* $p = 1-q$ *the boundary of the metric compactification is*

$$\{-\frac{q}{1-q}|y|^{1-q}\}.$$

(2) *when* $q \in]0, 1/2]$ *and* $p > 1-q$, *or when* $q \in]1/2, 1[$, $p \geq (2-q)/3$ *the boundary of the metric compactification is* $\{0\}$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^n$. As in $\S 2.1$ we note that, if $x_n \neq 0$, we have

$$d_{p,q}(x_n, 0) = \frac{|x_n|}{|x_n|^q} = |x_n|^{1-q}.$$

If $q = 1$ this quotient is one, but if $q \in]0, 1[$ then $d_{p,q}(x_n, 0)$ diverge to $+\infty$ if, and only if $|x_n|$ diverge to $+\infty$. We have:

$$h_{x_n}(y) = d_{p,q}(x_n, y) - d_{p,q}(x_n, 0) = \frac{|x_n - y|}{\left(\sqrt[p]{|x_n|^p + |y|^p}\right)^q} - |x_n|^{1-q}.$$

In order to compute the limit of this last expression, we define

$$A = |x_n - y|, \quad B = |x_n|^p + |y|^p \text{ and } C = |x_n|^{1-q}.$$

Although A, B and C depend on n, for simplicity the dependence will not be made explicit. Clearly,

$$h_{x_n}(y) = \frac{A}{B^{q/p}} - C = \left(\frac{A}{B^{q/p}} - C\right)\frac{\frac{A}{B^{q/p}} + C}{\frac{A}{B^{q/p}} + C} = \frac{\frac{A^2}{B^{2q/p}} - C^2}{\frac{A}{B^{q/p}} + C}.$$

Let us evaluate the asymptotic behaviour of both numerator and denominator. Starting by the denominator:

$$\frac{A}{B^{q/p}} + C = \frac{|x_n - y|}{\left(\sqrt[p]{|x_n|^p + |y|^p}\right)^q} + |x_n|^{1-q} = |x_n|^{1-q}\left(\frac{\left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right|}{\left(\sqrt[p]{1 + \frac{|y|^p}{|x_n|^p}}\right)^q} + 1\right).$$

Recall that

$$1 - \left|\frac{y}{x_n}\right| \leq \left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right| \leq 1 + \left|\frac{y}{x_n}\right|.$$

Now, we need the dependency on n. Defining $D_n = \left(\sqrt[p]{1+\frac{|y|^p}{|x_n|^p}}\right)^q$ we can write

$$\frac{1}{D_n} - \frac{1}{D_n}\left|\frac{y}{x_n}\right| \leq \frac{1}{D_n}\left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right| \leq \frac{1}{D_n} + \frac{1}{D_n}\left|\frac{y}{x_n}\right|,$$

and

$$|x_n|^{1-q}\left(\frac{1}{D_n} + 1 - \frac{1}{D_n}\left|\frac{y}{x_n}\right|\right) \leq \frac{A}{B^{q/p}} + C \leq |x_n|^{1-q}\left(\frac{1}{D_n} + 1 + \frac{1}{D_n}\left|\frac{y}{x_n}\right|\right).$$

This leads to

$$|x_n|^{1-q}\left(\frac{1}{D_n} + 1\right) - \frac{1}{D_n}\frac{|y|}{|x_n|^q} \leq \frac{A}{B^{q/p}} + C \leq |x_n|^{1-q}\left(\frac{1}{D_n} + 1\right) + \frac{1}{D_n}\frac{|y|}{|x_n|^q}.$$

As n goes to infinity $D_n \to 1$ and $|x_n|^q \to +\infty$ and we see that

$$\frac{A}{B^{q/p}} + C$$

has the asymptotic behavior of $2|x_n|^{1-q}$, i.e.,

$$\frac{A}{B^{q/p}} + C \sim 2|x_n|^{1-q}$$

Regarding the numerator:

$$\begin{aligned}
\frac{A^2}{B^{2q/p}} - C^2 = \quad & \frac{A^2 - C^2 B^{2q/p}}{B^{2q/p}} = \frac{|x_n - y|^2 - |x_n|^{2-2q}\left(\sqrt[p]{|x_n|^p + |y|^p}\right)^{2q}}{\left(\sqrt[p]{|x_n|^p + |y|^p}\right)^{2q}} \\
= \quad & |x_n|^{2-2q}\left(\frac{\left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right|^2 - \left(\sqrt[p]{1+\frac{|y|^p}{|x_n|^p}}\right)^{2q}}{\left(\sqrt[p]{1+\frac{|y|^p}{|x_n|^p}}\right)^{2q}}\right).
\end{aligned}$$

For the root in the numerator of this expression we will use the first-order Taylor expansion

$$(1+t)^\alpha = 1 + \alpha t + o(t), (t \to 0).$$

Now, similarly to the last case, we can write

$$\left(1 - \left|\frac{y}{x_n}\right|\right)^2 \leq \left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right|^2 \leq \left(1 + \left|\frac{y}{x_n}\right|\right)^2,$$

Therefore, subtracting $\left(\sqrt[p]{1+\frac{|y|^p}{|x_n|^p}}\right)^{2q}$ gives, on the LHS of the inequality,

$$\left|\frac{y}{x_n}\right|^2 - 2\left|\frac{y}{x_n}\right| - \frac{2q}{p}\left|\frac{y}{x_n}\right|^p + \epsilon_n \leq \left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right|^2 - \left(\sqrt[p]{1+\frac{|y|^p}{|x_n|^p}}\right)^{2q},$$

and, on the RHS of the inequality,

$$\left|\frac{x_n}{|x_n|} - \frac{y}{|x_n|}\right|^2 - \left(\sqrt[p]{1+\frac{|y|^p}{|x_n|^p}}\right)^{2q} \leq \left|\frac{y}{x_n}\right|^2 + 2\left|\frac{y}{x_n}\right| - \frac{2q}{p}\left|\frac{y}{x_n}\right|^p + \epsilon_n$$

where $\epsilon_n = |y|^p \times o\Big(\dfrac{1}{|x_n|^p}\Big)$.

Hence, we obtain on the LHS,

$$\frac{|y|^2}{|x_n|^{2q}} - \frac{2|y|}{|x_n|^{2q-1}} - \frac{2q}{p}\frac{|y|^p}{|x_n|^{p+2q-2}} + \frac{\epsilon_n}{|x_n|^{2q-2}} \le \frac{A^2}{B^{2q/p}} - C^2,$$

and, on the RHS,

$$\frac{A^2}{B^{2q/p}} - C^2 \le \frac{|y|^2}{|x_n|^{2q}} + \frac{2|y|}{|x_n|^{2q-1}} - \frac{2q}{p}\frac{|y|^p}{|x_n|^{p+2q-2}} + \frac{\epsilon_n}{|x_n|^{2q-2}}.$$

Finally, using the fact that the denominator is asymptotic to $2|x_n|^{1-q}$, we get

$$\frac{|y|^2}{2|x_n|^{1+q}} - \frac{|y|}{|x_n|^q} - \frac{q}{p}\frac{|y|^p}{|x_n|^{p+q-1}} + \frac{\epsilon_n}{2|x_n|^{q-1}} \le h_{x_n}(y)$$

and

$$h_{x_n}(y) \le \frac{|y|^2}{2|x_n|^{1+q}} + \frac{|y|}{|x_n|^q} - \frac{q}{p}\frac{|y|^p}{|x_n|^{p+q-1}} + \frac{\epsilon_n}{2|x_n|^{q-1}}.$$

Since q is greater than zero the terms with denominators with exponents q and $1+q$ go to zero as $|x_n| \to +\infty$. If we choose p, q such that $p+q=1$, then

$$\frac{\epsilon_n}{2|x_n|^{q-1}} = \frac{|y|^p}{2} \times \frac{o\Big(\dfrac{1}{|x_n|^{1-q}}\Big)}{\dfrac{1}{|x_n|^{1-q}}} \to 0.$$

It follows that

$$\lim_{x_n \to +\infty} h_{x_n}(y) = -\frac{q}{1-q}|y|^{1-q}.$$

Recall that $d_{p,q}(x,y)$ is a metric for $q \in]0,1]$ and

$$p \ge \max\{1-q, (2-q)/3\}$$

By inspection we can see that if $q \in]0, 1/2]$ the condition on p is $p \ge 1-q$. This is compatible with $p+q=1$. If $q \in]1/2, 1[$ the condition on p is $p \ge (2-q)/3$, but now this condition is incompatible with $p+q=1$. Summarizing, for the metric space $(\mathbb{R}^n, d_{1-q,q})$ with $q \in]0,1/2]$ we have the metric functional

$$h(y) = -\frac{q}{1-q}|y|^{1-q}.$$

If $q \in]0,1/2]$ and $p > 1-q$ or $q \in]1/2, 1[$ and $p \ge (2-q)/3$, the metric functional is the null function. ∎

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