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Master in Financial Mathematics

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Resumo

Esta dissertação tem como objetivo principal explorar e analisar em detalhe o modelo de dois fatores proposto por Eraker e Wu (2017) em diferentes contextos financeiros. Inicialmente, apresentamos o modelo e sua configuração de equilíbrio sob a medida física. Detalhamos a especificação completa do modelo, incluindo os processos estocásticos envolvidos, e representamos as suas equações em notação matricial para facilitar a análise. Além disso, estudamos transformações afins que levam a uma simplificação e melhor compreensão do modelo. Posto isto, estendemos a nossa análise ao considerar o mesmo modelo de dois fatores, mas sob a medida de risco neutro. Introduzimos o conceito de fator de desconto estocástico, que é fundamental para avaliar os ativos financeiros, quando trabalhamos com a medida de risco neutro. Novamente, detalhamos a especificação do modelo sob esta medida, mantendo uma abordagem rigorosa e discutindo as transformações afins envolvidas que simplificam a análise. Feita toda a análise, concentramos a nossa atenção no prémio associado aos contratos de futuros do índice VIX (ao quadrado), um conceito importante no que diz respeito à volatilidade dos mercados. Utilizando equações e resultados derivados nos capítulos anteriores, exploramos de maneira aprofundada alguns fundamentos subjacentes a este prémio, sendo o principal, demonstrar que estes contratos têm retornos esperados negativos e como podem ser aplicados na gestão de risco e estratégias de investimento.

Abstract

This dissertation has as its main objective to explore and analyze in detail the two-factor model proposed by Eraker and Wu (2017) in different financial contexts. Initially, we show the model and its equilibrium specification under physical measure. We detail the full specification of the model, including the stochastic processes that are involved, and represent the equations in matricial notation to facilitate the analysis. Moreover, we study affine transformations, which allow us to simplify and better understand of the model. That said, we extend our analysis to consider the same two-factor model, but now under the risk-neutral measure. We then introduce stochastic discount factor concept, that is fundamental to evaluate financial assets, when working on risk-neutral measure. Again, the model specification is detailed under this measure, keeping a rigorous approach and affine transforms are proposed to simplify the analysis. Later, we focus our attention in the premium associated with futures contracts of the VIX (-squared) index, an important concept as far as the volatility market is concerned. Using equations and results derived on previous chapters, we explore in depth some underlying fundamentals to this premium, being the main one, to prove that these contracts have negative expected values and how they can be applied in risk management and investment strategies.

Contents

CHAPTER 1

Introduction

Futures tied to the VIX volatility index made their debut in the financial markets through the efforts of the Chicago Board Options Exchange (CBOE). Initially, their primary purpose was to gauge market fear (Bekaert and Hoerova (2014)) Bekaert, Hoerova 2014, and in their early days, these contracts were relatively unpopular among investors due to a lack of understanding. However, as time passed, they gradually gained liquidity and garnered greater interest from a wider range of investors (Eraker and Wu (2017)) Eraker, Wu 2017.

Hence, the popularity of investing in VIX futures contracts experienced a significant increase, primarily fueled by the advantages they offer in terms of portfolio diversification. Understanding the risks associated with financial markets is vital for investors and analysts. One of the instruments used to evaluate market volatility is the VIX(-squared) futures contracts. This contract has been the subject of increasing interest in the financial literature too, especially with regard to the risk premium associated with it. These contracts are utilized as a means to hedge against market downturns and volatility spikes, effectively safeguarding portfolios during unforeseen periods. Notably, these contracts display a negative correlation with the performance of the Standard & Poor's (S&P) 500 (Alexander and Korovilas (2012)) Alexander, Korovilas 2012, rendering them an appealing choice for risk management and hedging strategies.

So since the VIX has become substantially more volatile than the S&P 500, with a CAPM-based analysis of the index and futures on this, we get negative market betas (Eraker and Wu (2017)) Eraker, Wu 2017.

The main goal of this thesis is to explain the negative returns on the VIX futures. In other words, is it expected that in a long term position on the VIX futures, we get losses at maturity date 'T'?

The intuition behind the negative premium is the investor's willingness to pay in order to avoid market volatility, even if they can probably lose that amount. This phenomenon attracted the attention of researchers and raised questions about how the risk premium behaves in the face of the increase in overall market risk. Previous studies, suggest that the negative values of the VIX premium tends to "fall or stay flat when risk rises" (Cheng (2019)) Cheng 2019.

Interestingly, at first glance, it would be reasonable to assume that any premium linked to volatility should increase as the overall market risk increases. However, this assumption raises additional questions about the underlying factors that may influence the risk premiums associated with volatility.

CHAPTER 2

Literature Review

The comprehension of financial models is fundamental to make decisions in investment contexts and risk management. In this way, academic research is exploring a large variety of approaches to modulating and analyse dynamics of financial assets. An important model is the two-factor model proposed by Eraker and Wu (2017) Eraker, Wu 2017. This model offers a structure that can capture crucial volatility events and evaluate financial assets associated to them.

The objective of our dissertation is to show that VIX(-squared) futures have negative annualized returns, which is assumed by Alexander and Korovilas (2012) Alexander, Korovilas 2012. To do that, we start to state equations, with help of Duffie, Pan and Singleton (2000) Duffie, Pan, Singleton 2000, firstly in physical measure and then in the risk-neutral measure, so that we can output the equilibrium stock market price. Still within this theme, we will deduce a closed-form solution for the stochastic discount factor, that together with Eraker and Shaliastovich (2008) Eraker, Shaliastovich 2008, helps in the study of the risk-neutral measure. Martin (2017) Martin 2017, offers a closed-form for the VIX(-squared) and then, for the expected risk premium. We show, using Eraker and Wu (2017) Eraker, Wu 2017, that the risk premium must be negative.

CHAPTER 3

Two-factor Eraker and Wu (2017) model under the physical measure

We start by defining the two-factor model suggested by Eraker and Wu, under the physical measure. A measure of physical risk seeks to assess the inherent risk in financial assets based on the observation of real market prices. This approach presupposes that prices adjust in accordance with fluctuations observed in the market, thus reflecting a genuine value of the asset in question. It relies on the analysis of volatility and the historical behavior of prices, thereby providing a robust assessment of the risk associated with investments and financial assets.

3.1. Model specification

Let x be a random walk, which is a stochastic process that represents the stock market prices. We assume that $\tilde{x_T}$ is the terminal value of x, and $\tilde{x_T} = x_T$. σ_t is the volatility of process, which is driven by a Brownian motion, B_t^v, and a compound Poisson process, $\xi_t dN_t$ where $\{N_u; u \in [t, T]\}$ has Poisson arrivals with intensity l_0. Doing the same as Eraker and Wu (2017) Eraker, Wu 2017, at date t the cash flow of x_t satisfies Heston (1993) Heston 1993 model:

$$\frac{dx_t}{x_t} = \mu dt + \sigma_t dB_t^x, \tag{3.1}$$

where $\mu \in \mathbb{R}$ and B_t^x is a standard Brownian motion defined under the physical measure $\mathbb{P}$.

The instantaneous variance of asset returns (σ_t^2) is assumed to follow a square root process with jumps:

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \sigma_v \sigma_t dB_t^v + \xi_t dN_t, \tag{3.2}$$

where $\kappa \geq 0$ is the speed of mean reversion, θ is the long term mean of the process and σ_v is the volatility of the variance process. These parameters are assumed to satisfy the Feller condition:

$$\frac{2\kappa\theta}{\sigma_v^2} \geq 1.$$

The remaining parameters must adhere to the following conditions:

$$\xi_t \sim exp(\mu_\xi) \tag{3.3}$$

$$d\langle B^x, B^v\rangle_t = 0 \tag{3.4}$$

$$d\langle B^x, N\rangle_t = 0 \tag{3.5}$$

$$d\langle B^v, N\rangle_t = 0. \tag{3.6}$$

Note that, in this setup, the cash flow shocks (equation (3.1)) have stochastic volatility. So, using Itô's lemma,

$$\begin{aligned} \mathrm{d}\ lnx_t &= \frac{\partial}{\partial x_t}(ln\ x_t)\ \mathrm{d}x_t + \frac{1}{2}\frac{\partial^2}{\partial x_t^2}(ln\ x_t)d\langle x, x\rangle_t \\ &= \frac{1}{x_t}\left(x_t\mu dt + x_t\sigma_t dB_t^x\right) + \frac{1}{2}\left(-\frac{1}{x_t^2}\right)\left(x_t^2\sigma_t^2\mathrm{d}t\right) \\ &= \left(\mu - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t dB_t^x. \end{aligned} \tag{3.7}$$

Definition 3.1. *The marginal utility function is defined by Eraker and Wu (2017) Eraker, Wu 2017 as*

$$u'(x) = x^{-\gamma}. \tag{3.8}$$

Proposition 3.1. *The power utility function is*

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}. \tag{3.9}$$

Proof.

$$u(x) = \int u'(x)dx = \int x^{-\gamma}dx = \frac{x^{1-\gamma}}{1-\gamma}$$

yielding a power utility function:

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}.$$

□

3.1.1. Model in matricial notation

In this section, we introduce on matricial notation the two-factor model proposed by Eraker and Wu (2017) Eraker, Wu 2017 under physical measure. The matricial notation simplify the equations and make easier the analysis in general. We built matricial notation from the equations presented in the previous section.

Defining $X_t := \begin{bmatrix} ln\ x_t & \sigma_t^2 \end{bmatrix}'$, then equations (3.2) and (3.7) yield

$$dX_t = \begin{bmatrix} \mathrm{d}\ ln\ x_t \\ \mathrm{d}\ \sigma_t^2 \end{bmatrix} = \begin{bmatrix} \mu - \frac{1}{2}\sigma_t^2 \\ \kappa\left(\theta - \sigma_t^2\right) \end{bmatrix} dt + \begin{bmatrix} \sigma_t & 0 \\ 0 & \sigma_v \sigma_t \end{bmatrix} \cdot \begin{bmatrix} dB_t^x \\ dB_t^v \end{bmatrix} + \begin{bmatrix} 0 \\ \xi_t dN_t \end{bmatrix}$$

Since $\begin{bmatrix} \mu - \frac{1}{2}\sigma_t^2 \\ \kappa\left(\theta - \sigma_t^2\right) \end{bmatrix} = \begin{bmatrix} \mu \\ \kappa\theta \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa \end{bmatrix} \cdot \begin{bmatrix} ln\ x_t \\ \sigma_t^2 \end{bmatrix}$, we can write

$$dX_t = \left(\kappa_0 + \kappa_1 \cdot X_t\right) dt + \sigma(X_t) \cdot \begin{bmatrix} dB_t^x \\ dB_t^v \end{bmatrix} + \begin{bmatrix} 0 \\ \xi_t dN_t \end{bmatrix}, \tag{3.10}$$

where

$$\kappa_0 = \begin{bmatrix} \mu \\ \kappa\theta \end{bmatrix}, \tag{3.11}$$

$$\kappa_1 = \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa \end{bmatrix}, \tag{3.12}$$

$$\sigma(X_t) = \begin{bmatrix} \sigma_t & 0 \\ 0 & \sigma_v \sigma_t \end{bmatrix}. \tag{3.13}$$

Moreover,

$$\sigma(X_t) \cdot \sigma(X_t)' = H_0 + H \cdot X_t = \begin{bmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_v^2 \sigma_t^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \times ln\ x_t + \begin{bmatrix} 1 & 0 \\ 0 & \sigma_v^2 \end{bmatrix} \times \sigma_t^2, \tag{3.14}$$

where $H = \begin{bmatrix} H_1 & H_2 \end{bmatrix}$, $H_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $H_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $H_2 = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$.

3.2. Affine transform

Now, we go deeper on our research, and begin the study of two crucial functions that will help, in the course of work.

Following Duffie, Pan and Singleton (2000, Proposition 1) Duffie, Pan, Singleton 2000, for $u \in \mathbb{R}^2$

$$\Psi(u, X_t; t, T) := \mathbb{E}_{\mathbb{P}}\left[e^{u' \cdot X_T} \mid \mathcal{F}_t\right]$$

$$= \exp\left[\alpha(u; t, T) + \beta'(u; t, T) \cdot X_t\right], \tag{3.15}$$

where $\beta(u; t, T)$ and $\alpha(u; t, T)$ solve the real-valued ordinary differential equations

$$\frac{\partial}{\partial t}\beta(u; t, T) = -\kappa_1' \cdot \beta(u; t, T) - \frac{1}{2}\begin{bmatrix}\beta'(u; t, T) \cdot H_1 \cdot \beta(u; t, T)\\ \beta'(u; t, T) \cdot H_2 \cdot \beta(u; t, T)\end{bmatrix}, \tag{3.16}$$

and

$$\frac{\partial}{\partial t}\alpha(u; t, T) = -\kappa_0' \cdot \beta(u; t, T) - \frac{1}{2}\beta'(u; t, T) \cdot H_0 \cdot \beta(u; t, T) - l_0\left[\rho\left(\beta_2\left(u; t, T\right)\right) - 1\right], \tag{3.17}$$

with

$$\rho(h) = \frac{1}{1 - \mu_\xi h}, \tag{3.18}$$

subject to the boundary conditions

$$\alpha(u; T, T) = 0 \tag{3.19}$$

and

$$\beta(u; T, T) = u. \tag{3.20}$$

Combining equations (3.12), (3.14) and (3.16), then

$$\frac{\partial \beta}{\partial t}(u;t,T) = -\begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & -\kappa \end{bmatrix} \cdot \begin{bmatrix} \beta_1(u;t,T) \\ \beta_2(u;t,T) \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ \beta_1^2(u;t,T) + \sigma_v^2 \beta_2^2(u;t,T) \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} \frac{\partial \beta_1}{\partial t}(u;t,T) \\ \frac{\partial \beta_2}{\partial t}(u;t,T) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}\beta_1(u;t,T) + \kappa\beta_2(u;t,T) - \frac{1}{2}\beta_1^2(u;t,T) - \frac{1}{2}\sigma_v^2\beta_2^2(u;t,T) \end{bmatrix},$$

yielding

$$\beta_1(u;t,T) = u_1, \tag{3.21}$$

and

$$\frac{\partial \beta_2}{\partial t}(u;t,T) = \frac{1}{2}u_1 + \kappa\beta_2(u;t,T) - \frac{1}{2}u_1^2 - \frac{1}{2}\sigma_v^2\beta_2^2(u;t,T).$$

Proposition 3.2. *The second component of the vector β is given by*

$$\beta_2(u;t,T) = \frac{\kappa - d(u_1) - (\kappa + d(u_1))\, q(u) e^{-d(u_1)\tau}}{\sigma_v^2 \left(1 - q(u) e^{-d(u_1)\tau}\right)}, \tag{3.22}$$

where

$$d(u_1) = \sqrt{\kappa^2 + \sigma_v^2 u_1 (1 - u_1)}, \tag{3.23}$$

and

$$q(u) = \frac{\kappa - d(u_1) - u_2\sigma_v^2}{\kappa + d(u_1) - u_2\sigma_v^2}, \tag{3.24}$$

with

$$\tau := T - t. \tag{3.25}$$

Proof. Let's start by factoring the equation of the β_2.

$$\begin{aligned} \frac{\partial \beta_2}{\partial t}(u;t,T) &= \frac{1}{2}u_1 + \kappa\beta_2(u;t,T) - \frac{1}{2}u_1^2 - \frac{1}{2}\sigma_v^2\beta_2^2(u;t,T) \\ &= -\frac{1}{2}\sigma_v^2\beta_2^2(u;t,T) + \kappa\beta_2(u;t,T) + \frac{1}{2}u_1\left(1 - u_1\right). \end{aligned}$$

Factorizing the equation above,

$$-\frac{1}{2}\sigma_v^2\beta_2^2(u;t,T)+\kappa\beta_2(u;t,T)+\frac{1}{2}u_1\left(1-u_1\right)=0$$

and then solving in order to find $\beta_2(u;t,T)$,

$$\beta_2(u;t,T)=\frac{-\kappa\pm\sqrt{\kappa^2-4\left(-\frac{1}{2}\sigma_v^2\right)\left(\frac{1}{2}u_1\left(1-u_1\right)\right)}}{2\left(-\frac{1}{2}\right)\sigma_v^2}$$

$$=\frac{\kappa\pm\sqrt{\kappa^2+\sigma_v^2u_1\left(1-u_1\right)}}{\sigma_v^2}.$$

Define

$$d(u_1)=\sqrt{\kappa^2+\sigma_v^2u_1\left(1-u_1\right)}.$$

In this way,

$$-\frac{1}{2}\sigma_v^2\beta_2^2(u;t,T)+\kappa\beta_2(u;t,T)+\frac{1}{2}u_1\left(1-u_1\right)$$

can be rewritten as

$$-\frac{1}{2}\sigma_v^2\left(\beta_2(u;t,T)-\frac{\kappa+d(u_1)}{\sigma_v^2}\right)\left(\beta_2(u;t,T)-\frac{\kappa-d(u_1)}{\sigma_v^2}\right).$$

Therefore, the expression for $\frac{\partial\beta_2}{\partial t}(u;t,T)$ is the same,

$$\frac{\partial\beta_2}{\partial t}(u;t,T)=-\frac{1}{2}\sigma_v^2\left(\beta_2(u;t,T)-\frac{\kappa+d(u_1)}{\sigma_v^2}\right)\left(\beta_2(u;t,T)-\frac{\kappa-d(u_1)}{\sigma_v^2}\right)$$

and, hence,

$$\frac{1}{-\frac{1}{2}\sigma_v^2}\frac{1}{\left(\beta_2(u;t,T)-\frac{\kappa+d(u_1)}{\sigma_v^2}\right)\left(\beta_2(u;t,T)-\frac{\kappa-d(u_1)}{\sigma_v^2}\right)}\partial\beta_2(u;t,T)=\partial t$$

Putting the fraction as sum of simply fractions, we can solve the problem, and then appears the question: For what values of a and b, the following sentence was satisfying?

$$\frac{\partial\beta_2(u;t,T)}{-\frac{1}{2}\sigma_v^2}\left(\frac{a}{\beta_2(u;t,T)-\frac{\kappa+d(u_1)}{\sigma_v^2}}+\frac{b}{\beta_2(u;t,T)-\frac{\kappa-d(u_1)}{\sigma_v^2}}\right)=\partial t$$

We know that,

$$1 = \left(\beta_2(u;t,T) - \frac{\kappa - d(u_1)}{\sigma_v^2}\right) a + \left(\beta_2(u;t,T) - \frac{\kappa + d(u_1)}{\sigma_v^2}\right) b.$$

Therefore,

$$\begin{cases} 1 = -\frac{\kappa - d(u_1)}{\sigma_v^2} a - \frac{\kappa + d(u_1)}{\sigma_v^2} b \\ 0 = a\beta_2(u;t,T) + b\beta_2(u;t,T) \end{cases}$$

The second equation produce $a = -b$.

The first one yields,

$$1 = \frac{\kappa - d(u_1)}{\sigma_v^2} b - \frac{\kappa + d(u_1)}{\sigma_v^2} b$$

$$= \frac{-d(u_1)}{\sigma_v^2} b - \frac{d(u_1)}{\sigma_v^2} b = \frac{-2d(u_1)}{\sigma_v^2} b$$

Solving in order to b,

$$b = -\frac{\sigma_v^2}{2d(u_1)}$$

and, hence,

$$a = \frac{\sigma_v^2}{2d(u_1)}.$$

Going back,

$$\frac{\partial \beta_2(u;t,T)}{-\frac{1}{2}\sigma_v^2} \left(\frac{\frac{\sigma_v^2}{2d(u_1)}}{\beta_2(u;t,T) - \frac{\kappa + d(u_1)}{\sigma_v^2}} + \frac{-\frac{\sigma_v^2}{2d(u_1)}}{\beta_2(u;t,T) - \frac{\kappa - d(u_1)}{\sigma_v^2}} \right) = \partial t$$

$$-\frac{\sigma_v^2}{-\frac{1}{2}\sigma_v^2 2d(u_1)} \left(\frac{\partial \beta_2(u;t,T)}{\beta_2(u;t,T) - \frac{\kappa - d(u_1)}{\sigma_v^2}} - \frac{\partial \beta_2(u;t,T)}{\beta_2(u;t,T) - \frac{\kappa + d(u_1)}{\sigma_v^2}} \right) = \partial t$$

$$\frac{1}{d(u_1)} \left(\frac{\partial \beta_2(u;t,T)}{\beta_2(u;t,T) - \frac{\kappa - d(u_1)}{\sigma_v^2}} - \frac{\partial \beta_2(u;t,T)}{\beta_2(u;t,T) - \frac{\kappa + d(u_1)}{\sigma_v^2}} \right) = \partial t$$

$$\frac{\partial \beta_2(u;t,T)}{\beta_2(u;t,T) - \frac{\kappa - d(u_1)}{\sigma_v^2}} - \frac{\partial \beta_2(u;t,T)}{\beta_2(u;t,T) - \frac{\kappa + d(u_1)}{\sigma_v^2}} = d(u_1)\, \partial t.$$

Integrate both sides:

$$ln\left(\beta_2(u;t,T)-\frac{\kappa-d(u_1)}{\sigma_v^2}\right)-ln\left(\beta_2(u;t,T)-\frac{\kappa+d(u_1)}{\sigma_v^2}\right)=d(u_1)t+C$$

$$ln\left(\frac{\beta_2(u;t,T)-\frac{\kappa-d(u_1)}{\sigma_v^2}}{\beta_2(u;t,T)-\frac{\kappa+d(u_1)}{\sigma_v^2}}\right)=d(u_1)t+C$$

$$\frac{\beta_2(u;t,T)-\frac{\kappa-d(u_1)}{\sigma_v^2}}{\beta_2(u;t,T)-\frac{\kappa+d(u_1)}{\sigma_v^2}}=e^{d(u_1)t}e^C.$$

Regrouping the terms to find $\beta_2(u;t,T)$:

$$\beta_2(u;t,T)-\frac{\kappa-d(u_1)}{\sigma_v^2}=e^{d(u_1)t}e^C\left(\beta_2(u;t,T)-\frac{\kappa+d(u_1)}{\sigma_v^2}\right)$$

$$\beta_2(u;t,T)\left(1-e^{d(u_1)t}e^C\right)=\frac{\kappa-d(u_1)}{\sigma_v^2}-\frac{\kappa+d(u_1)}{\sigma_v^2}e^{d(u_1)t}e^C$$

$$\beta_2(u;t,T)=\frac{\kappa-d(u_1)-(\kappa+d(u_1))\,e^{d(u_1)t}e^C}{\sigma_v^2\left(1-e^{d(u_1)t}e^C\right)}. \tag{3.26}$$

As we know that $\beta_2(u;T,T)=u_2$,

$$u_2=\frac{\kappa-d(u_1)-(\kappa+d(u_1))\,e^{d(u_1)T}e^C}{\sigma_v^2\left(1-e^{d(u_1)T}e^C\right)}$$

$$u_2\sigma_v^2\left(1-e^{d(u_1)T}e^C\right)=\kappa-d(u_1)-(\kappa+d(u_1))\,e^{d(u_1)T}e^C$$

$$\left(\kappa+d(u_1)-u_2\sigma_v^2\right)e^{d(u_1)T}e^C=\kappa-d(u_1)-u_2\sigma_v^2.$$

Therefore,

$$e^C=\frac{\kappa-d(u_1)-u_2\sigma_v^2}{\kappa+d(u_1)-u_2\sigma_v^2}e^{-d(u_1)T}.$$

Defining

$$q(u)=\frac{\kappa-d(u_1)-u_2\sigma_v^2}{\kappa+d(u_1)-u_2\sigma_v^2}$$

then,

$$e^{C} = q(u)e^{-d(u_1)T}. \tag{3.27}$$

Finally, combining the equations (3.23) - (3.27):

$$\beta_2(u;t,T) = \frac{\kappa - d(u_1) - (\kappa + d(u_1))\, q(u)e^{-d(u_1)(T-t)}}{\sigma_v^2 \left(1 - q(u)e^{-d(u_1)(T-t)}\right)}.$$

□

In summary, and combining equations (3.21) and (3.22), we have:

$$\beta(u;t,T) = \begin{bmatrix} u_1 \\ \frac{\kappa - d(u_1) - (\kappa + d(u_1))q(u)e^{-d(u_1)(T-t)}}{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-t)}\right)} \end{bmatrix} \tag{3.28}$$

and, hence,

$$\lim_{T \to +\infty} \beta(u;t,T) = \begin{bmatrix} u_1 \\ \frac{\kappa - d(u_1)}{\sigma_v^2} \end{bmatrix}. \tag{3.29}$$

Concerning the function $\alpha(u;t,T)$:

Proposition 3.3. *Under the two-factor Eraker and Wu (2017) model,*

$$\begin{aligned} \alpha(u;t,T) = &\left(\mu u_1 - l_0 + \frac{\kappa\theta}{\sigma_v^2}\left(\kappa - d(u_1)\right) + \frac{\sigma_v^2 l_0}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\right)\tau \\ &+ \frac{2\kappa\theta}{\sigma_v^2} \times ln\left(\frac{1 - q(u)}{1 - q(u)e^{-d(u_1)\tau}}\right) \\ &- \frac{2\sigma_v^2 \mu_\xi l_0}{\left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right)\left(\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)\right)} \\ &\times ln\left(\frac{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u)}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u)e^{-d(u_1)\tau}}\right). \end{aligned} \tag{3.30}$$

Proof.

$$
\begin{aligned}
\frac{\partial \alpha}{\partial t}(u;t,T) &= -\kappa_0' \cdot \beta(u;t,T) - \frac{1}{2}\beta'(u;t,T)\cdot H_0 \cdot \beta(u;t,T) - l_0\left[\rho\left(\beta_2\left(u;t,T\right)\right)-1\right] \\
&= -\kappa_0' \cdot \beta(u;t,T) - l_0\left[\rho\left(\beta_2\left(u;t,T\right)\right)-1\right] \\
&= -\begin{bmatrix}\mu & \kappa\theta\end{bmatrix} \cdot \begin{bmatrix}\beta_1(u;t,T)\\ \beta_2(u;t,T)\end{bmatrix} - l_0\left(\frac{1}{1-\mu_\xi\beta_2(u;t,T)}-1\right) \\
&= -\mu\beta_1(u;t,T) - \kappa\theta\beta_2(u;t,T) - l_0\left(\frac{1}{1-\mu_\xi\beta_2(u;t,T)}-1\right) \\
&= -\mu u_1 - \kappa\theta\beta_2(u;t,T) - l_0\left(\frac{1}{1-\mu_\xi\beta_2(u;t,T)}-1\right).
\end{aligned}
\tag{3.31}
$$

Now, that we have a more simplified expression for the $\alpha(u;t,T)$, we integrate from t to T:

$$
\int_t^T \frac{\partial \alpha}{\partial t}(u;s,T)\,ds = \alpha(u;T,T) - \alpha(u;t,T) = -\alpha(u;t,T)
$$

and therefore

$$
\alpha(u;t,T) = -\int_t^T \frac{\partial \alpha}{\partial t}(u;s,T)\,ds.
$$

Thus, replacing in the above equation, the expression of $\frac{\partial \alpha}{\partial t}(u;s,T)$:

$$
\begin{aligned}
\alpha(u;t,T) &= \int_t^T \mu u_1 + \kappa\theta\beta_2(u;s,T) + l_0\left(\frac{1}{1-\mu_\xi\beta_2(u;s,T)}-1\right)ds \\
&= \int_t^T \mu u_1 + \kappa\theta\beta_2(u;s,T) + l_0\left(\frac{1}{1-\mu_\xi\beta_2(u;s,T)}\right) - l_0\,ds \\
&= \left(\mu u_1 - l_0\right)\tau + \kappa\theta\int_t^T \beta_2(u;s,T)\,ds + l_0\int_t^T \frac{1}{1-\mu_\xi\beta_2(u;s,T)}\,ds.
\end{aligned}
\tag{3.32}
$$

Solving the following integrals (Corollary 3.1. and Corollary 3.2.), we achieve a closed-form for $\alpha(u;t,T)$:

$$\begin{aligned}\alpha(u;t,T) = &\left(\mu u_1 - l_0 + \frac{\kappa\theta}{\sigma_v^2}\left(\kappa - d(u_1)\right) + \frac{\sigma_v^2 l_0}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\right)\tau \\ &+ \frac{2\kappa\theta}{\sigma_v^2} \times ln\left(\frac{1-q(u)}{1-q(u)e^{-d(u_1)\tau}}\right) \\ &- \frac{2\sigma_v^2\mu_\xi l_0}{\left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right)\left(\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)\right)} \\ &\times ln\left(\frac{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right)q(u)}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right)q(u)e^{-d(u_1)\tau}}\right).\end{aligned}$$

□

Let's state two corollaries, each one dealing with integrals separately, then apply to equation (3.32) and conclude.

Corollary 3.1. *Under the two-factor Eraker and Wu (2017) model,*

$$\int_t^T \beta_2(u;s,T)\,ds = \frac{(\kappa - d(u_1))\,\tau + 2ln\left(\frac{1-q(u)}{1-q(u)e^{-d(u_1)\tau}}\right)}{\sigma_v^2}. \tag{3.33}$$

Proof.

$$\int_t^T \beta_2(u;s,T)\,ds$$

$$= \int_t^T \frac{\kappa - d(u_1) - (\kappa + d(u_1))\,q(u)e^{-d(u_1)(T-s)}}{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right)}\,ds$$

$$= \int_t^T \frac{\kappa - d(u_1) - (\kappa - d(u_1))\,q(u)e^{-d(u_1)(T-s)} - 2q(u)d(u_1)e^{-d(u_1)(T-s)}}{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right)}\,ds$$

$$= \frac{1}{\sigma_v^2}\int_t^T \frac{(\kappa - d(u_1))\left[1 - q(u)e^{-d(u_1)(T-s)}\right] - 2q(u)d(u_1)e^{-d(u_1)(T-s)}}{\left(1 - q(u)e^{-d(u_1)(T-s)}\right)}\,ds$$

$$= \frac{1}{\sigma_v^2}\int_t^T \kappa - d(u_1) + \frac{-2q(u)d(u_1)e^{-d(u_1)(T-s)}}{1 - q(u)e^{-d(u_1)(T-s)}}\,ds$$

$$= \frac{1}{\sigma_v^2}\left[(\kappa - d(u_1))\,\tau + 2\int_t^T \frac{-q(u)d(u_1)e^{-d(u_1)(T-s)}}{1 - q(u)e^{-d(u_1)(T-s)}}\,ds\right]$$

$$= \frac{\kappa - d(u_1)}{\sigma_v^2}\tau + \frac{2}{\sigma_v^2}\left[ln\left(1 - q(u)e^{-d(u_1)(T-T)}\right) - ln\left(1 - q(u)e^{-d(u_1)(T-t)}\right)\right]$$

$$= \frac{\kappa - d(u_1)}{\sigma_v^2}\tau + \frac{2}{\sigma_v^2}ln\left(\frac{1 - q(u)}{1 - q(u)e^{-d(u_1)(T-t)}}\right)$$

$$= \frac{(\kappa - d(u_1))\,\tau + 2ln\left(\frac{1-q(u)}{1-q(u)e^{-d(u_1)\tau}}\right)}{\sigma_v^2}.$$

□

Corollary 3.2. *Under the two-factor Eraker and Wu (2017) model,*

$$\int_t^T \frac{1}{1-\mu_\xi \beta_2(u;s,T)}\, ds = \frac{\sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\,\tau$$
$$+ \frac{-\sigma_v^2 2\mu_\xi}{\left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right)\left(\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)\right)}$$
$$\times\ ln\left(\frac{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u)}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) e^{-d(u_1)\tau}}\right). \tag{3.34}$$

Proof.

$$\frac{1}{1-\mu_\xi \beta_2(u;s,T)}$$

$$= \frac{1}{1-\mu_\xi \frac{\kappa - d(u_1) - (\kappa + d(u_1)) q(u) e^{-d(u_1)(T-s)}}{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right)}}$$

$$= \frac{1}{\frac{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right) - \mu_\xi\left(\kappa - d(u_1) - (\kappa + d(u_1)) q(u) e^{-d(u_1)(T-s)}\right)}{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right)}}$$

$$= \frac{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right)}{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right) - \mu_\xi\left(\kappa - d(u_1) - \left(\kappa + d(u_1)\right) q(u) e^{-d(u_1)(T-s)}\right)}$$

$$= \frac{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right)}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) e^{-d(u_1)(T-s)}}. \tag{3.35}$$

Separating equation (3.35) in sum of two terms, one of them its independent of s,

$$a + \frac{b e^{-d(u_1)(T-s)}}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) e^{-d(u_1)(T-s)}}.$$

But, for what values of a and b, the expression makes sense?
We know that,

$$\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right)$$

$$= a\left[\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)\right] + a\left[\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right] q(u) e^{-d(u_1)(T-s)} + b e^{-d(u_1)(T-s)}.$$

Therefore,

$$\begin{cases} \sigma_v^2 = a\left[\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)\right] \\ -\sigma_v^2 q(u) = a\left[\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right] q(u) + b \end{cases}$$

In the first equation we isolate the variable a and in the second equation we isolate the variable b.

$$\begin{cases} a = \frac{\sigma_v^2}{\sigma_v^2 - \mu_\xi(\kappa - d(u_1))} \\ b = -\sigma_v^2 q(u) - a\left[\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right] q(u) \end{cases}.$$

In this way, there is a closed-form for a, so replacing in the second equation to get a closed-form for b:

$$b = -\sigma_v^2 q(u) - \frac{\sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\left[\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right] q(u)$$

$$= -\sigma_v^2 q(u)\left(1 + \frac{\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\right)$$

$$= -\sigma_v^2 q(u)\left(\frac{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)} + \frac{\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\right)$$

$$= -\sigma_v^2 q(u)\left(\frac{2\mu_\xi d(u_1)}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\right).$$

Back to equation (3.35), we can rewrite as following:

$$\frac{\sigma_v^2\left(1 - q(u)e^{-d(u_1)(T-s)}\right)}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u)e^{-d(u_1)(T-s)}}$$

$$= \frac{\sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)} + \frac{-\sigma_v^2 q(u)\left(\frac{2\mu_\xi d(u_1)}{\sigma_v^2 - \mu_\xi(\kappa - d(u_1))}\right) e^{-d(u_1)(T-s)}}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u)e^{-d(u_1)(T-s)}}.$$

So,

$$\int_t^T \frac{1}{1-\mu_\xi \beta_2(u;s,T)}\, ds$$

$$= \int_t^T \frac{\sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)} - \frac{\sigma_v^2 q(u)\left(\frac{2\mu_\xi d(u_1)}{\sigma_v^2 - \mu_\xi(\kappa - d(u_1))}\right) e^{-d(u_1)(T-s)}}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) e^{-d(u_1)(T-s)}}\, ds$$

$$= \frac{\sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\tau - \int_t^T \frac{\sigma_v^2 q(u)\left(\frac{2\mu_\xi d(u_1)}{\sigma_v^2 - \mu_\xi(\kappa - d(u_1))}\right) e^{-d(u_1)(T-s)}}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) e^{-d(u_1)(T-s)}}\, ds$$

$$= \frac{\sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\tau - \frac{\sigma_v^2 q(u)\left(\frac{2\mu_\xi d(u_1)}{\sigma_v^2 - \mu_\xi(\kappa - d(u_1))}\right)}{\left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) d(u_1)} \times \int_t^T \frac{\left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) d(u_1) e^{-d(u_1)(T-s)}}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) e^{-d(u_1)(T-s)}}\, ds$$

$$= \frac{\sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\tau - \frac{\sigma_v^2 \frac{2\mu_\xi}{\sigma_v^2 - \mu_\xi(\kappa - d(u_1))}}{\left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right)} \times ln\left(\frac{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) e^{-d(u_1)(T-T)}}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) e^{-d(u_1)(T-t)}}\right)$$

$$= \frac{\sigma_v^2}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)}\tau - \frac{\sigma_v^2 2\mu_\xi}{\left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right)\left(\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right)\right)} \times ln\left(\frac{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u)}{\sigma_v^2 - \mu_\xi\left(\kappa - d(u_1)\right) + \left(\mu_\xi\left(\kappa + d(u_1)\right) - \sigma_v^2\right) q(u) e^{-d(u_1)\tau}}\right).$$

□

3.3. Equilibrium stock market price

The equilibrium asset price can be found by solving the optimal portfolio problem (Eraker and Wu, Appendix A3) Eraker, Wu 2017:

$$\max_s \mathbb{E}_t\left[u\left(s\tilde{x}_T - (s-1) P_t e^{r_f(T-t)}\right)\right], \tag{3.36}$$

where s is the number of shares held by the representative agent, and P_t is the price of the risky asset at date t.

From the first order condition, we take the derivative of equation (3.36) with respect to s, and set it equal to zero, yielding

$$\mathbb{E}_t\left[\left(\tilde{x}_T - P_t e^{r_f(T-t)}\right) u'\left(s\tilde{x}_T - (s-1)\, P_t e^{r_f(T-t)}\right)\right] = 0.$$

With fixed supply $s^* = 1$,

$$\mathbb{E}_t\left[\tilde{x}_T u'\left(\tilde{x}_T\right) - P_t e^{r_f(T-t)} u'\left(\tilde{x}_T\right)\right] = 0,$$

i.e.,

$$\mathbb{E}_t\left[\tilde{x}_T u'\left(\tilde{x}_T\right)\right] - P_t e^{r_f(T-t)} \mathbb{E}_t\left[u'\left(\tilde{x}_T\right)\right] = 0,$$

and, hence,

$$P_t = \frac{\mathbb{E}_t^{\mathbb{P}}\left[\tilde{x}_T \cdot u'\left(\tilde{x}_T\right)\right]}{\mathbb{E}_t^{\mathbb{P}}\left[u'\left(\tilde{x}_T\right)\right] e^{r_f \tau}}. \tag{3.37}$$

Using equation (3.8), and equation (3.37) becomes

$$\begin{aligned} P_t &= \frac{\mathbb{E}_t\left[\tilde{x}_T^{1-\gamma}\right]}{\mathbb{E}_t\left[\tilde{x}_T^{-\gamma}\right] e^{r_f \tau}} \\ &= \frac{\mathbb{E}_t\left[exp\left((1-\gamma)\, ln\ \tilde{x}_T\right)\right]}{\mathbb{E}_t\left[exp\left(-\gamma ln\ \tilde{x}_T\right)\right] e^{r_f \tau}} \\ &= \frac{\mathbb{E}_t\left[exp\left((1-\gamma)\, ln\ \tilde{x}_T + 0 \times \sigma_t^2\right)\right]}{\mathbb{E}_t\left[exp\left(-\gamma ln\ \tilde{x}_T + 0 \times \sigma_t^2\right)\right] e^{r_f \tau}} \\ &= \frac{\Psi\left(\begin{bmatrix} 1-\gamma & 0 \end{bmatrix}, X_t; t, T\right)}{\Psi\left(\begin{bmatrix} -\gamma & 0 \end{bmatrix}, X_t; t, T\right) e^{r_f \tau}}, \end{aligned} \tag{3.38}$$

where the last equality comes from the equation (3.15).

Defining

$$u_{1-\gamma} = \begin{bmatrix} 1-\gamma & 0 \end{bmatrix} \tag{3.39}$$

and

$$u_{-\gamma} = \begin{bmatrix} -\gamma & 0 \end{bmatrix}, \tag{3.40}$$

we can rewrite equation (3.38) as

$$P_t = \frac{\Psi\left(u_{1-\gamma}, X_t; t, T\right)}{\Psi\left(u_{-\gamma}, X_t; t, T\right) e^{r_f \tau}}$$

$$= \frac{exp\left[\alpha\left(u_{1-\gamma}; t, T\right) + \beta'\left(u_{1-\gamma}; t, T\right) X_t\right]}{exp\left[\alpha\left(u_{-\gamma}; t, T\right) + \beta'\left(u_{-\gamma}; t, T\right) X_t\right] e^{r_f \tau}}$$

$$= e^{-r_f \tau + (\alpha(u_{1-\gamma}; t,T) - \alpha(u_{-\gamma}; t,T)) + ((1-\gamma)-(-\gamma)) ln\ x_t + (\beta_2(u_{1-\gamma}; t,T) - \beta_2(u_{-\gamma}; t,T))\sigma_t^2}. \tag{3.41}$$

Defining

$$\lambda_0(t, T) = \alpha\left(u_{1-\gamma}; t, T\right) - \alpha\left(u_{-\gamma}; t, T\right), \tag{3.42}$$

and

$$\lambda_\sigma(t, T) = \beta_2\left(u_{1-\gamma}; t, T\right) - \beta_2\left(u_{-\gamma}; t, T\right) \tag{3.43}$$

then equation (3.41) can be simplified into

$$P_t = e^{-r_f \tau + \lambda_0(t,T) + ln\ x_t + \lambda_\sigma(t,T)\sigma_t^2}$$

$$= x_t e^{-r_f \tau + \lambda_0(t,T) + \lambda_\sigma(t,T)\sigma_t^2}. \tag{3.44}$$

The equilibrium stock price is given by

$$\mathrm{d}P_t = \mathrm{d}x_t e^{-r_f \tau + \lambda_0(t,T) + \lambda_\sigma(t,T)\sigma_t^2}$$
$$+ x_t \left[\left(r_f + \frac{\partial \lambda_0(t,T)}{\partial t} + \frac{\partial \lambda_\sigma(t,T)}{\partial t}\sigma_t^2\right)\mathrm{d}t + \lambda_\sigma(t,T)\mathrm{d}\sigma_t^2\right] e^{-r_f \tau + \lambda_0(t,T) + \lambda_\sigma(t,T)\sigma_t^2}$$

$$= \mathrm{d}x_t \frac{P_t}{x_t} + P_t\left[\left(r_f + \frac{\partial \lambda_0(t,T)}{\partial t} + \frac{\partial \lambda_\sigma(t,T)}{\partial t}\sigma_t^2\right)\mathrm{d}t + \lambda_\sigma(t,T)\mathrm{d}\sigma_t^2\right],$$

i.e.,

$$\frac{\mathrm{d}P_t}{P_t} = \frac{\mathrm{d}x_t}{x_t} + \left(r_f + \frac{\partial\lambda_0(t,T)}{\partial t} + \frac{\partial\lambda_\sigma(t,T)}{\partial t}\sigma_t^2\right)\mathrm{d}t + \lambda_\sigma(t,T)\mathrm{d}\sigma_t^2,$$

and, hence,

$$\mathrm{d}lnP_t = \mathrm{d}lnx_t + r_f\mathrm{d}t + \frac{\partial\lambda_0(t,T)}{\partial t}\mathrm{d}t + \frac{\partial\lambda_\sigma(t,T)}{\partial t}\sigma_t^2\mathrm{d}t + \lambda_\sigma(t,T)\mathrm{d}\sigma_t^2. \tag{3.45}$$

In order to reduce the effects of time passing, we have chosen to approach the model by considering an infinite time horizon, thus addressing the issue in a more comprehensive manner.

PROPOSITION 3.4. *The equilibrium stock price can be expressed as*

$$lnP_T = lnP_t - \lambda_x' \cdot X_t + (r_f + \lambda_0)\,\tau + \lambda_x' \cdot X_T, \tag{3.46}$$

where

$$\lambda_x' := \begin{bmatrix} 1 & \lambda_\sigma \end{bmatrix}, \tag{3.47}$$

$$\lambda_0 := \lim_{T\to+\infty} \frac{\partial\lambda_0(t,T)}{\partial t} \tag{3.48}$$

and

$$\lambda_\sigma := \lim_{T\to\infty} \lambda_\sigma(t,T). \tag{3.49}$$

PROOF. To establish this, we are systematically examining the matter by progressively extending the parameter "T" towards infinity (referred to as the infinite horizon limit), and then performing integration over the interval from t to T.

1.

$$\frac{\partial \lambda_0(t,T)}{\partial t} = \frac{\partial}{\partial t}\alpha\left(u_{1-\gamma};t,T\right) - \frac{\partial}{\partial t}\alpha\left(u_{-\gamma};t,T\right)$$

$$\begin{aligned}
&= -\begin{bmatrix}\mu & \kappa\theta\end{bmatrix}\cdot\begin{bmatrix}\beta_1(u_{1-\gamma})\\ \beta_2(u_{1-\gamma})\end{bmatrix} - l_0\left(\frac{1}{1-\mu_\xi\beta_2(u_{1-\gamma})}-1\right)\\
&\quad -\left(-\begin{bmatrix}\mu & \kappa\theta\end{bmatrix}\cdot\begin{bmatrix}\beta_1(u_{-\gamma})\\ \beta_2(u_{-\gamma})\end{bmatrix} - l_0\left(\frac{1}{1-\mu_\xi\beta_2(u_{-\gamma})}-1\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= -\mu\left(1-\gamma\right) - \kappa\theta\beta_2(u_{1-\gamma}) - l_0\left(\frac{1}{1-\mu_\xi\beta_2(u_{1-\gamma})}-1\right)\\
&\quad + \mu\left(-\gamma\right) + \kappa\theta\beta_2(u_{-\gamma}) + l_0\left(\frac{1}{1-\mu_\xi\beta_2(u_{-\gamma})}-1\right)
\end{aligned}$$

$$= -\mu - \kappa\theta\left(\beta_2(u_{1-\gamma}) - \beta_2(u_{-\gamma})\right) - l_0\left(\frac{1}{1-\mu_\xi\beta_2(u_{1-\gamma})} - \frac{1}{1-\mu_\xi\beta_2(u_{-\gamma})}\right)$$

Define

$$\lambda_0 = \lim_{T\to+\infty}\frac{\partial \lambda_0(t,T)}{\partial t}.$$

2.

$$\frac{\partial \lambda_\sigma(t,T)}{\partial t} = \frac{\partial}{\partial t}\beta_2\left(u_{1-\gamma};t,T\right) - \frac{\partial}{\partial t}\beta_2\left(u_{-\gamma};t,T\right)$$

$$\begin{aligned}
&= \frac{1}{2}\left(1-\gamma\right)\left(1-(1-\gamma)\right) + \kappa\beta_2(u_{1-\gamma}) - \frac{1}{2}\sigma_v^2\beta_2^2(u_{1-\gamma})\\
&\quad -\left(\frac{1}{2}\left(-\gamma\right)\left(1-(-\gamma)\right) + \kappa\beta_2(u_{-\gamma}) - \frac{1}{2}\sigma_v^2\beta_2^2(u_{-\gamma})\right)
\end{aligned}$$

$$= \frac{1}{2}\left(1-\gamma\right)\gamma + \frac{1}{2}\left(1+\gamma\right)\gamma + \kappa\left(\beta_2(u_{1-\gamma}) - \beta_2(u_{-\gamma})\right) - \frac{1}{2}\sigma_v^2\left(\beta_2^2(u_{1-\gamma}) - \beta_2^2(u_{-\gamma})\right)$$

$$= \frac{1}{2}\gamma\left(1-\gamma+1+\gamma\right) + \kappa\left(\beta_2(u_{1-\gamma}) - \beta_2(u_{-\gamma})\right) - \frac{1}{2}\sigma_v^2\left(\beta_2^2(u_{1-\gamma}) - \beta_2^2(u_{-\gamma})\right)$$

$$= \gamma + \kappa\left(\beta_2(u_{1-\gamma}) - \beta_2(u_{-\gamma})\right) - \frac{1}{2}\sigma_v^2\left(\beta_2^2(u_{1-\gamma}) - \beta_2^2(u_{-\gamma})\right).$$

When we take the limit,

$$\lim_{T\to+\infty} \beta_2(u_{1-\gamma}) = \frac{\kappa - d(1-\gamma)}{\sigma_v^2}$$

and

$$\lim_{T\to+\infty} \beta_2(u_{-\gamma}) = \frac{\kappa - d(-\gamma)}{\sigma_v^2}.$$

2.1.

$$\lim_{T\to\infty} \frac{\partial \lambda_\sigma(t,T)}{\partial t}$$

$$= \gamma + \kappa \left(\frac{\kappa - d(1-\gamma)}{\sigma_v^2} - \frac{\kappa - d(-\gamma)}{\sigma_v^2} \right) - \frac{1}{2}\sigma_v^2 \left(\frac{\kappa^2 - 2\kappa d(1-\gamma) + d^2(1-\gamma)}{\sigma_v^4} - \frac{\kappa^2 - 2\kappa d(-\gamma) + d^2(-\gamma)}{\sigma_v^4} \right)$$

$$= \gamma + \frac{\kappa}{\sigma_v^2} \left(d(-\gamma) - d(1-\gamma) \right) - \frac{1}{2} \left(\frac{-2\kappa d(1-\gamma) + d^2(1-\gamma) + 2\kappa d(-\gamma) - d^2(-\gamma)}{\sigma_v^2} \right)$$

$$= \gamma - \frac{1}{2\sigma_v^2} \left(d^2(1-\gamma) - d^2(-\gamma) \right)$$

$$= \gamma - \frac{1}{2\sigma_v^2} \left[\left(\kappa^2 + \sigma_v^2(1-\gamma)(1-(1-\gamma)) \right) - \left(\kappa^2 + \sigma_v^2(-\gamma)(1-(-\gamma)) \right) \right]$$

$$= \gamma - \frac{1}{2\sigma_v^2} \left[\kappa^2 + \sigma_v^2(1-\gamma)\gamma - \kappa^2 - \sigma_v^2(-\gamma)(1+\gamma) \right]$$

$$= \gamma - \frac{1}{2\sigma_v^2} \left[\sigma_v^2(1-\gamma)\gamma + \sigma_v^2\gamma(1+\gamma) \right]$$

$$= \gamma - \frac{1}{2}\gamma \left(1 - \gamma + 1 + \gamma \right)$$

$$= \gamma - \frac{2\gamma}{2} = \gamma - \gamma = 0. \tag{3.50}$$

3.

$$\lim_{T\to\infty}\lambda_\sigma(t,T) = \lim_{T\to\infty}\beta_2(u_{1-\gamma};t,T) - \lim_{T\to\infty}\beta_2(u_{-\gamma};t,T)$$

$$= \frac{\kappa - d(1-\gamma)}{\sigma_v^2} - \frac{\kappa - d(-\gamma)}{\sigma_v^2}$$

$$= \frac{d(-\gamma) - d(1-\gamma)}{\sigma_v^2} = \lambda_\sigma. \tag{3.51}$$

Back to the equation (3.45) and taking $T \to \infty$,

$$\mathrm{d}lnP_t = \mathrm{d}lnx_t + r_f\mathrm{d}t + \lim_{T\to\infty}\frac{\partial\lambda_0(t,T)}{\partial t}\mathrm{d}t + \lim_{T\to\infty}\frac{\partial\lambda_\sigma(t,T)}{\partial t}\sigma_t^2\mathrm{d}t + \lim_{T\to\infty}\lambda_\sigma(t,T)\mathrm{d}\sigma_t^2.$$

Combining equations (3.48), (3.50) and (3.51)

$$\mathrm{d}lnP_t = \mathrm{d}lnx_t + r_f\mathrm{d}t + \lambda_0\mathrm{d}t + \lambda_\sigma\mathrm{d}\sigma_t^2. \tag{3.52}$$

Using equations (3.7), (3.48) and (3.49):

$$\begin{aligned}\mathrm{d}lnP_t = &\left(\mu - \frac{1}{2}\sigma_t^2\right)\mathrm{d}t + \sigma_t\mathrm{d}B_t^x + r_f\mathrm{d}t + \lambda_\sigma\mathrm{d}\sigma_t^2 \\ &+ \left[-\mu - \kappa\theta\lambda_\sigma - l_0\left(\rho\left(\beta_2\left(u_{1-\gamma};\infty\right)\right) - \rho\left(\beta_2\left(u_{-\gamma};\infty\right)\right)\right)\right]\mathrm{d}t \\ = &\, r_f\mathrm{d}t + \sigma_t\mathrm{d}B_t^x + \lambda_\sigma\mathrm{d}\sigma_t^2 + \lambda_0(\sigma_t^2)\mathrm{d}t\end{aligned}$$

that is equation (11) of Eraker and Wu Eraker, Wu 2017, where

$$\lambda_0(\sigma_t^2) = -\frac{1}{2}\sigma_t^2 - \kappa\theta\lambda_\sigma - l_0\left(\rho\left(\beta_2\left(u_{1-\gamma}\right)\right) - \rho\left(\beta_2\left(u_{-\gamma}\right)\right)\right). \tag{3.53}$$

Integrating equation (3.52) from t to T,

$$\int_t^T \mathrm{d}lnP_u = \int_t^T \mathrm{d}lnx_u + \int_t^T r_f \ \mathrm{d}u + \int_t^T \lambda_0 \ \mathrm{d}u + \int_t^T \lambda_\sigma \ \mathrm{d}\sigma_u^2$$

which yields,

$$lnP_T - lnP_t = lnx_T - lnx_t + r_f\tau + \lambda_0\tau + \lambda_\sigma\sigma_T^2 - \lambda_\sigma\sigma_t^2.$$

Passing lnP_t to the right-hand side and reassembling the terms,

$$\begin{aligned} lnP_T &= lnP_t - \left(lnx_t + \lambda_\sigma\sigma_t^2\right) + (r_f + \lambda_0)\,\tau + \left(lnx_T + \lambda_\sigma\sigma_T^2\right) \\ &= lnP_t - \begin{bmatrix}1 & \lambda_\sigma\end{bmatrix} \cdot \begin{bmatrix} lnx_t \\ \sigma_t^2 \end{bmatrix} + (r_f + \lambda_0)\,\tau + \begin{bmatrix}1 & \lambda_\sigma\end{bmatrix} \cdot \begin{bmatrix} lnx_T \\ \sigma_T^2 \end{bmatrix} \\ &= lnP_t - \lambda_x' \cdot X_t + (r_f + \lambda_0)\,\tau + \lambda_x' \cdot X_T, \end{aligned}$$

where

$$\lambda_x' = \begin{bmatrix}1 & \lambda_\sigma\end{bmatrix}.$$

□

CHAPTER 4

Two-factor Eraker and Wu (2017) model under the risk-neutral measure

In this chapter, we will analise the two-factor Eraker and Wu (2017) model under the risk-neutral measure.

4.1. Stochastic discount factor

We present the mathematical formulation of the stochastic discount factor within the framework of the two-factor Eraker and Wu (2017) model Eraker, Wu 2017 exploring its practical applications.

We know from equation (3.38) that the price can be written as

$$
\begin{aligned}
P_t &= \frac{\mathbb{E}_t\left[\tilde{x}_T^{1-\gamma}\right]}{\mathbb{E}_t\left[\tilde{x}_T^{-\gamma}\right] \times e^{r_f \tau}} \\
&= \frac{\mathbb{E}_t\left[\tilde{x}_T^{1-\gamma} \times e^{-r_f T}\right]}{\mathbb{E}_t\left[\tilde{x}_T^{-\gamma} \times e^{-r_f t}\right]} \\
&= \mathbb{E}_t\left[\frac{\mathbb{E}_T\left(\tilde{x}_T^{1-\gamma} \times e^{-r_f T}\right)}{\mathbb{E}_t\left(\tilde{x}_T^{-\gamma} \times e^{-r_f t}\right)}\right] \\
&= \mathbb{E}_t\left[\frac{\mathbb{E}_T\left(\tilde{x}_T^{1-\gamma} \times e^{-r_f T}\right)}{\mathbb{E}_t\left(\tilde{x}_T^{-\gamma} \times e^{-r_f t}\right)} \times \frac{\mathbb{E}_T\left(\tilde{x}_T^{-\gamma} \times e^{-r_f T}\right)}{\mathbb{E}_T\left(\tilde{x}_T^{-\gamma} \times e^{-r_f T}\right)}\right] \\
&= \mathbb{E}_t\left[\frac{\mathbb{E}_T\left(\tilde{x}_T^{-\gamma} \times e^{-r_f T}\right)}{\mathbb{E}_t\left(\tilde{x}_T^{-\gamma} \times e^{-r_f t}\right)} \times \frac{\mathbb{E}_T\left(\tilde{x}_T^{1-\gamma} \times e^{-r_f T}\right)}{\mathbb{E}_T\left(\tilde{x}_T^{-\gamma} \times e^{-r_f T}\right)}\right] \\
&= \mathbb{E}_t\left[\frac{\mathbb{E}_T\left(\tilde{x}_T^{-\gamma} \times e^{-r_f T}\right)}{\mathbb{E}_t\left(\tilde{x}_T^{-\gamma} \times e^{-r_f t}\right)} \times P_T\right]. \qquad (4.1)
\end{aligned}
$$

The term that multiplies P_T, on the equation above is called the stochastic discount factor $\frac{M_T}{M_t}$, and, hence,

$$P_t = \mathbb{E}_t\left[\frac{M_T}{M_t} \times P_T\right].$$

PROPOSITION 4.1. *Under the two-factor Eraker and Wu (2017) model, M_t can be expressed as*

$$M_t = e^{\alpha(u_{-\gamma},t,T)+\beta'(u_{-\gamma},t,T)X_t - r_f t}. \tag{4.2}$$

PROOF.

$$M_t = \mathbb{E}_t\left(\tilde{x}_T^{-\gamma} \times e^{-r_f t}\right)$$

$$= \mathbb{E}_t\left(\tilde{x}_T^{-\gamma}\right) \times e^{-r_f t}$$

$$= \mathbb{E}_t\left(e^{-\gamma \times ln\tilde{x}_T}\right) \times e^{-r_f t}$$

$$= \mathbb{E}_t\left(e^{-\gamma \times ln\tilde{x}_T + 0\times\sigma_T^2}\right) \times e^{-r_f t}$$

$$= \mathbb{E}_t\left(e^{u_{-\gamma}\cdot X_T}\right) \times e^{-r_f t}$$

and now using the information of equation (3.15):

$$M_t = e^{\alpha(u_{-\gamma},t,T)+\beta'(u_{-\gamma},t,T)X_t} \times e^{-r_f t}$$

$$= e^{\alpha(u_{-\gamma},t,T)+\beta'(u_{-\gamma},t,T)X_t - r_f t}.$$

□

Applying the same approach as in the equilibrium stock price to obtain equation (3.45), differentiation of equation (4.2) yields

$$dM_t = \left(\alpha\left(u_{-\gamma};t,T\right) + \beta'\left(u_{-\gamma};t,T\right)\cdot X_t - r_f t\right)dt \times M_t$$

then,

$$\frac{dM_t}{M_t} = \left(\alpha\left(u_{-\gamma};t,T\right) + \beta_1\left(u_{-\gamma};t,T\right)ln\ x_t + \beta_2\left(u_{-\gamma};t,T\right)\sigma_t^2 - r_f t\right)dt$$

$$= \left(\alpha\left(u_{-\gamma};t,T\right) - \gamma ln\ x_t + \beta_2\left(u_{-\gamma};t,T\right)\sigma_t^2 - r_f t\right)dt$$

$$= \frac{\partial}{\partial t}\alpha\left(u_{-\gamma};t,T\right)dt - \gamma dln\ x_t + \frac{\partial}{\partial t}\beta_2\left(u_{-\gamma};t,T\right)\sigma_t^2 dt + \beta_2\left(u_{-\gamma};t,T\right)d\sigma_t^2 - r_f dt.$$

Therefore,

$$
\begin{aligned}
dlnM_t = & -\gamma dln\ x_t - r_f dt + \frac{\partial}{\partial t}\alpha\left(u_{-\gamma};t,T\right)dt \\
& + \frac{\partial}{\partial t}\beta_2\left(u_{-\gamma};t,T\right)\sigma_t^2 dt + \beta_2\left(u_{-\gamma};t,T\right)d\sigma_t^2.
\end{aligned} \tag{4.3}
$$

Now, just as we did in Chapter 3, let's apply the infinite horizon limit to mitigate the impacts of the time lapsing.

1.

$$\lim_{T\to\infty}\beta_2(u_{-\gamma};t,T) = \frac{\kappa - d(-\gamma)}{\sigma_v^2}.$$

2.

$$
\begin{aligned}
\lim_{T\to\infty}\frac{\partial}{\partial t}\beta_2(u_{-\gamma};t,T) &= \lim_{T\to\infty}\left(-\frac{1}{2}\sigma_v^2\beta_2^2 + \kappa\beta_2 - \frac{1}{2}\gamma\left(1+\gamma\right)\right) \\
&= -\frac{1}{2}\sigma_v^2\left(\frac{\kappa^2 - 2\kappa d(-\gamma) + d^2(-\gamma)}{\sigma_v^4}\right) + \kappa\frac{\kappa - d(-\gamma)}{\sigma_v^2} - \frac{\gamma\left(1+\gamma\right)}{2} \\
&= -\frac{1}{2}\frac{\kappa^2 - 2\kappa d(-\gamma) + d^2(-\gamma)}{\sigma_v^2} + \frac{\kappa^2 - \kappa d(-\gamma)}{\sigma_v^2} - \frac{\gamma\left(1+\gamma\right)}{2} \\
&= \frac{\kappa^2 - d^2(-\gamma)}{2\sigma_v^2} - \frac{\gamma\left(1+\gamma\right)}{2}.
\end{aligned}
$$

Using equation (3.23):

$$
\begin{aligned}
\lim_{T\to\infty}\frac{\partial}{\partial t}\beta_2(u_{-\gamma};t,T) &= \frac{\kappa^2 - \left(\kappa^2 - \sigma_v^2\gamma\left(1+\gamma\right)\right)}{2\sigma_v^2} - \frac{\gamma\left(1+\gamma\right)}{2} \\
&= \frac{\gamma\left(1+\gamma\right)}{2} - \frac{\gamma\left(1+\gamma\right)}{2} = 0.
\end{aligned}
$$

3.

$$\lim_{T\to\infty} \frac{\partial}{\partial t}\alpha(u_{-\gamma}; t, T)$$

$$= \lim_{T\to\infty} \left(-\mu\beta_1(u_{-\gamma}; t, T) - \kappa\theta\beta_2(u_{-\gamma}; t, T) - l_0 \left(\frac{1}{1 - \mu_\xi \beta_2(u_{-\gamma}; t, T)} - 1 \right) \right)$$

$$= \mu\gamma - \kappa\theta \frac{\kappa - d(-\gamma)}{\sigma_v^2} - l_0 \left(\frac{1}{1 - \mu_\xi \frac{\kappa - d(-\gamma)}{\sigma_v^2}} - 1 \right) = \omega. \tag{4.4}$$

Now, by combining the previously calculated information, let's consolidate it all, resulting in the following:

$$dlnM_t = -\gamma \; dln \; x_t - r_f dt + \omega dt + \frac{\kappa - d(-\gamma)}{\sigma_v^2} d\sigma_t^2.$$

Integrating both sides from t to T, results in

$$\int_t^T dlnM_u = \int_t^T -\gamma \; dln \; x_u + \int_t^T \omega - r_f \; du + \int_t^T \frac{\kappa - d(-\gamma)}{\sigma_v^2} \; d\sigma_u^2,$$

i.e.,

$$lnM_T - lnM_t = -\gamma \left(lnx_T - lnx_t \right) + \left(\omega - r_f \right) \tau + \frac{\kappa - d(-\gamma)}{\sigma_v^2} \left(\sigma_T^2 - \sigma_t^2 \right).$$

Therefore,

$$lnM_T = lnM_t + \left(\gamma lnx_t - \frac{\kappa - d(-\gamma)}{\sigma_v^2} \sigma_t^2 \right) - \left(\gamma lnx_T - \frac{\kappa - d(-\gamma)}{\sigma_v^2} \sigma_T^2 \right) + \left(\omega - r_f \right) \tau$$

$$= lnM_t + \left[\gamma \quad -\tfrac{\kappa - d(-\gamma)}{\sigma_v^2} \right] \cdot X_t - \left[\gamma \quad -\tfrac{\kappa - d(-\gamma)}{\sigma_v^2} \right] \cdot X_T + \left(\omega - r_f \right) \tau$$

$$= lnM_t + \gamma_x' \cdot X_t - \gamma_x' \cdot X_T + \left(\omega - r_f \right) \tau, \tag{4.5}$$

where

$$\gamma_x' = \left[\gamma \quad -\tfrac{\kappa - d(-\gamma)}{\sigma_v^2} \right]. \tag{4.6}$$

4.2. Market Prices of Risk

The financial models used so far have a stochastic component that imparts randomness to asset prices, and this component is the diffusion term. Eraker and Wu (2017, equation (90)) Eraker, Wu 2017, define the diffusion term of stochastic process x as

$$
\begin{aligned}
&-\gamma\sigma_t dB_t^x - \eta\sigma_v\sigma_t dB_t^v && (4.7)\\
&= -\begin{bmatrix}\gamma\sigma_t & \eta\sigma_v\sigma_t\end{bmatrix}\cdot\begin{bmatrix}dB_t^x\\ dB_t^v\end{bmatrix}\\
&= -\begin{bmatrix}\gamma & \eta\end{bmatrix}\cdot\begin{bmatrix}\sigma_t & 0\\ 0 & \sigma_v\sigma_t\end{bmatrix}\cdot\begin{bmatrix}dB_t^x\\ dB_t^v\end{bmatrix}\\
&= -\begin{bmatrix}\gamma & \eta\end{bmatrix}\cdot\sigma(X_t)\cdot dB_t,
\end{aligned}
$$

with $\sigma(X_t) = \begin{bmatrix}\sigma_t & 0\\ 0 & \sigma_v\sigma_t\end{bmatrix}$ and $dB_t = \begin{bmatrix}dB_t^x\\ dB_t^v\end{bmatrix}$.

Since the diffusion terms are the same regardless of the authors, Eraker and Shaliastovich (2008, equation (2.22)) Eraker, Shaliastovich 2008 and Eraker and Wu(2017, equation (90)) Eraker, Wu 2017, this term are equal to $-\Lambda_t'\cdot dB_t$ with $\Lambda_t = \sigma(X_t)'\cdot\lambda$.

Thus,

$$-\lambda'\cdot\sigma(X_t)\cdot dB_t = -\begin{bmatrix}\gamma & \eta\end{bmatrix}\cdot\sigma(X_t)\cdot dB_t,$$

and, hence,

$$\lambda = \begin{bmatrix}\gamma & \eta\end{bmatrix}'. \tag{4.8}$$

4.3. Model Specification under the risk-neutral measure

Under the physical measure ($\mathbb{P}$), equations (3.10) - (3.14) can be stated as

$$dX_t = (\kappa_0 + \kappa_1\cdot X_t)\,dt + \sigma(X_t)\cdot dB_t + \begin{bmatrix}0\\ \xi_t dN_t\end{bmatrix},$$

with

$$\sigma(X_t)\cdot\sigma(X_t)' = H_0 + \sum_{i=1}^{n} H_i\cdot X_{t,i},$$

$$l(X_t) = l_0 + l_1\cdot X_t \text{ and}$$

$$\mathbb{E}\left(e^{\theta\xi_i}\right) = \rho\,(\theta).$$

Thus, under the risk-neutral measure ($\mathbb{Q}$), we have the same equation but with different terms,

$$dX_t = (\tilde{\kappa_0} + \tilde{\kappa_1} \cdot X_t)\, dt + \sigma(X_t) \cdot d\tilde{B}_t + \begin{bmatrix} 0 \\ \tilde{\xi}_t d\tilde{N}_t \end{bmatrix},$$

where

$$\tilde{\kappa_0} = \kappa_0 - H_0 \cdot \lambda, \tag{4.9}$$

$$\tilde{\kappa_1} = \kappa_1 - \begin{bmatrix} H_1\lambda & H_2\lambda & \cdots & H_n\lambda \end{bmatrix}, \tag{4.10}$$

$$\tilde{l}(X_t) = l(X_t) \cdot \rho\,(-\lambda) \tag{4.11}$$

and

$$\tilde{\rho}\,(u) = \rho\,(u - \lambda)\,./\rho\,(-\lambda) \tag{4.12}$$

using Eraker and Shaliastovich (2008, equations (2.24), (2.25), (2.27) and (2.28)) Eraker, Shaliastovich 2008.

Then, solving in order to find the terms under measure $\mathbb{Q}$,

$$\tilde{\kappa_0} = \kappa_0 - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \kappa_0 = \begin{bmatrix} \mu \\ \kappa\theta \end{bmatrix}, \tag{4.13}$$

$$\tilde{\kappa_1} = \kappa_1 - \begin{bmatrix} H_1\lambda & H_2\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa \end{bmatrix} - \begin{bmatrix} 0 & \gamma \\ 0 & \sigma_v^2\eta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2} - \gamma \\ 0 & -\kappa - \sigma_v^2\eta \end{bmatrix}$$

and

$$\kappa + \sigma_v^2\eta = \tilde{\kappa}. \tag{4.14}$$

Then,

$$\tilde{\kappa_1} = \begin{bmatrix} 0 & -\frac{1}{2} - \gamma \\ 0 & -\tilde{\kappa} \end{bmatrix}, \tag{4.15}$$

$$\tilde{l}(X_t) = \begin{bmatrix} 0 & 0 \\ 0 & l_0 \end{bmatrix} \cdot \begin{bmatrix} \rho(-\gamma) \\ \rho(-\eta) \end{bmatrix} = \begin{bmatrix} 0 \\ l_0 \rho(-\eta) \end{bmatrix} \tag{4.16}$$

with

$$l_0 \rho(-\eta) = \tilde{l_0} \tag{4.17}$$

and

$$\tilde{\rho}(u) = \begin{bmatrix} \rho(u_1 - \gamma) \\ \rho(u_2 - \eta) \end{bmatrix} ./ \begin{bmatrix} \rho(-\gamma) \\ \rho(-\eta) \end{bmatrix} = \begin{bmatrix} \rho(u_1 - \gamma)/\rho(-\gamma) \\ \rho(u_2 - \eta)/\rho(-\eta) \end{bmatrix}. \tag{4.18}$$

The second component of this vector is given by

$$\frac{\rho(u_2 - \eta)}{\rho(-\eta)} = \frac{1 + \mu_\xi \eta}{1 + \mu_\xi \eta - \mu_\xi u_2} = \frac{1}{1 - \frac{\mu_\xi}{1 + \mu_\xi \eta} u_2} = \tilde{\rho_2}(u_2),$$

where

$$\frac{\mu_\xi}{1 + \mu_\xi \eta} = \mu_\xi \times \frac{1}{1 - \mu_\xi(-\eta)} = \mu_\xi \rho(-\eta) = \tilde{\mu_\xi}. \tag{4.19}$$

Simplifying, the model under measure $\mathbb{Q}$ can be finally written as

$$dX_t = (\kappa_0 + \tilde{\kappa_1} \cdot X_t)\, dt + \sigma(X_t) \cdot d\tilde{B}_t + \begin{bmatrix} 0 \\ \tilde{\xi}_t d\tilde{N}_t \end{bmatrix}.$$

4.4. Affine Transform

As we did in the previous chapter, following Duffie et al. (2000) Duffie, Pan, Singleton 2000 and for $u \in \mathbb{R}^2$,

$$\Psi(u, t, T, X_t) = \mathbb{E}_{\mathbb{Q}} \left[e^{u' X_t} \mid \mathcal{F}_t \right]$$

$$= \exp\left[\tilde{\alpha}(u; t, T) + \tilde{\beta}'(u; t, T) \cdot X_t \right], \tag{4.20}$$

where $\tilde{\alpha}(u; t, T)$ and $\tilde{\beta}(u; t, T)$ solved the real valued ordinary differential equations

$$\frac{\partial}{\partial t} \tilde{\beta}(u; t, T) = -\tilde{\kappa}_1' \cdot \tilde{\beta}(u; t, T) - \frac{1}{2} \begin{bmatrix} \tilde{\beta}'(u; t, T) \cdot \tilde{H}_1 \cdot \tilde{\beta}(u; t, T) \\ \tilde{\beta}'(u; t, T) \cdot \tilde{H}_2 \cdot \tilde{\beta}(u; t, T) \end{bmatrix} \tag{4.21}$$

and

$$\frac{\partial}{\partial t}\tilde{\alpha}(u;t,T) = -\tilde{\kappa}_0' \cdot \tilde{\beta}(u;t,T) - \frac{1}{2}\tilde{\beta}'(u;t,T)\cdot \tilde{H}_0 \cdot \tilde{\beta}(u;t,T) - \tilde{l}_0\left[\tilde{\rho}\left(\tilde{\beta}_2\left(u;t,T\right)\right)-1\right], \quad (4.22)$$

subject to the boundary conditions

$$\tilde{\alpha}(u;T,T) = 0 \quad (4.23)$$

and

$$\tilde{\beta}(u;T,T) = u. \quad (4.24)$$

To solve for $\tilde{\beta}(u;t,T)$, let's remember equations for $\tilde{\kappa}_1, \tilde{H}_1$ and $\tilde{H}_2$:

$$\begin{bmatrix} \frac{\partial\tilde{\beta}_1}{\partial t}(u;t,T) \\ \frac{\partial\tilde{\beta}_2}{\partial t}(u;t,T) \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ -\frac{1}{2}-\gamma & -\tilde{\kappa} \end{bmatrix} \cdot \begin{bmatrix} \tilde{\beta}_1(u;t,T) \\ \tilde{\beta}_2(u;t,T) \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ \tilde{\beta}_1^2(u;t,T) + \sigma_v^2\tilde{\beta}_2^2(u;t,T) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \left(\frac{1}{2}+\gamma\right)\tilde{\beta}_1(u;t,T) + \tilde{\kappa}\tilde{\beta}_2(u;t,T) \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ \tilde{\beta}_1^2(u;t,T) + \sigma_v^2\tilde{\beta}_2^2(u;t,T) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \left(\frac{1}{2}+\gamma\right)\tilde{\beta}_1(u;t,T) + \tilde{\kappa}\tilde{\beta}_2(u;t,T) - \frac{1}{2}\tilde{\beta}_1^2(u;t,T) - \frac{1}{2}\sigma_v^2\tilde{\beta}_2^2(u;t,T) \end{bmatrix}. \quad (4.25)$$

Moreover,

$$\tilde{\beta}_1(u;t,T) = u_1 \quad (4.26)$$

and

$$\frac{\partial}{\partial t}\tilde{\beta}_2(u;t,T)$$

$$= \left(\frac{1}{2}+\gamma\right)\tilde{\beta}_1(u;t,T) + \tilde{\kappa}\tilde{\beta}_2(u;t,T) - \frac{1}{2}\tilde{\beta}_1^2(u;t,T) - \frac{1}{2}\sigma_v^2\tilde{\beta}_2^2(u;t,T)$$

$$= -\frac{1}{2}\sigma_v^2\tilde{\beta}_2^2(u;t,T) + \tilde{\kappa}\tilde{\beta}_2(u;t,T) + \frac{1}{2}\left(2\gamma+1-u_1\right)u_1. \quad (4.27)$$

Factorizing:

$$\frac{\partial}{\partial t}\tilde{\beta}_2(u;t,T) = 0$$

is equivalent to

$$-\frac{1}{2}\sigma_v^2\tilde{\beta}_2^2(u;t,T) + \tilde{\kappa}\tilde{\beta}_2(u;t,T) + \frac{1}{2}\left(2\gamma + 1 - u_1\right)u_1 = 0.$$

In this way,

$$\begin{aligned}
\tilde{\beta}_2(u;t,T) &= \frac{-\tilde{\kappa} \pm \sqrt{\tilde{\kappa}^2 - 4\left(-\frac{1}{2}\sigma_v^2\right)\left(\frac{1}{2}\left(2\gamma + 1 - u_1\right)u_1\right)}}{2\left(-\frac{1}{2}\right)\sigma_v^2} \\
&= \frac{-\tilde{\kappa} \pm \sqrt{\tilde{\kappa}^2 + \sigma_v^2\left(2\gamma + 1 - u_1\right)u_1}}{-\sigma_v^2} \\
&= \frac{\tilde{\kappa} \pm \sqrt{\tilde{\kappa}^2 + \sigma_v^2\left(2\gamma + 1 - u_1\right)u_1}}{\sigma_v^2}. \qquad (4.28)
\end{aligned}$$

Define

$$\tilde{d}(u_1) = \sqrt{\tilde{\kappa}^2 + \sigma_v^2\left(2\gamma + 1 - u_1\right)u_1}. \qquad (4.29)$$

If $\gamma = 0$, then $\tilde{d}(u_1) = d(u_1)$.

Thus,

$$\frac{\partial}{\partial t}\tilde{\beta}_2(u;t,T) = -\frac{1}{2}\sigma_v^2\left(\tilde{\beta}_2(u;t,T) - \frac{\tilde{\kappa} + \tilde{d}(u_1)}{\sigma_v^2}\right)\left(\tilde{\beta}_2(u;t,T) - \frac{\tilde{\kappa} - \tilde{d}(u_1)}{\sigma_v^2}\right).$$

Without loss of generality, repeating the same steps as we do in Section 3.2.1. to find $\beta(u;t,T)$, just replacing (κ, d, l, ρ) by $\left(\tilde{\kappa}, \tilde{d}, \tilde{l}, \tilde{\rho}\right)$:

$$\tilde{\beta}(u;t,T) = \begin{bmatrix} u_1 \\ \frac{\tilde{\kappa} - \tilde{d}(u_1) - \left(\tilde{\kappa} + \tilde{d}(u_1)\right)\tilde{q}(u)e^{-\tilde{d}(u_1)\tau}}{\sigma_v^2\left(1 - \tilde{q}(u)e^{-\tilde{d}(u_1)\tau}\right)} \end{bmatrix} \qquad (4.30)$$

where

$$\tilde{q}(u) = \frac{\tilde{\kappa} - \tilde{d}(u_1) - u_2\sigma_v^2}{\tilde{\kappa} + \tilde{d}(u_1) - u_2\sigma_v^2} \qquad (4.31)$$

and

$$\lim_{T\to+\infty}\tilde{\beta}(u;t,T) = \begin{bmatrix} u_1 \\ \frac{\tilde{\kappa}-\tilde{d}(u_1)}{\sigma_v^2} \end{bmatrix}. \tag{4.32}$$

Now, to solve the ordinary differential equation concerning $\tilde{\alpha}(u;t,T)$, we apply the same method. The ODE is

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{\alpha}(u;t,T) &= -\tilde{\kappa}_0' \cdot \tilde{\beta}(u;t,T) - \frac{1}{2}\tilde{\beta}'(u;t,T) \cdot \tilde{H}_0 \cdot \tilde{\beta}(u;t,T) - \tilde{l}_0 \left[\tilde{\rho}\left(\tilde{\beta}_2\left(u;t,T\right)\right) - 1\right] \\
&= -\tilde{\kappa}_0' \cdot \tilde{\beta}(u;t,T) - \tilde{l}_0 \left[\tilde{\rho}\left(\tilde{\beta}_2\left(u;t,T\right)\right) - 1\right] \\
&= -\tilde{\mu}\tilde{\beta}_1(u;t,T) - \tilde{\kappa}\tilde{\theta}\tilde{\beta}_2(u;t,T) - \tilde{l}_0 \left[\tilde{\rho}\left(\tilde{\beta}_2\left(u;t,T\right)\right) - 1\right].
\end{aligned}$$

Without loss of generality and similarly to what was done to find $\alpha(u;t,T)$ in Section 3.2.1., replacing again (κ, d, l, ρ) by $\left(\tilde{\kappa}, \tilde{d}, \tilde{l}, \tilde{\rho}\right)$:

$$\begin{aligned}
\tilde{\alpha}(u;t,T) = &\left(\tilde{\mu}u_1 - \tilde{l}_0 + \frac{\tilde{\kappa}\tilde{\theta}}{\sigma_v^2}\left(\tilde{\kappa} - \tilde{d}(u_1)\right) + \frac{\sigma_v^2\tilde{l}_0}{\sigma_v^2 - \tilde{\mu}_\xi\left(\tilde{\kappa} - \tilde{d}(u_1)\right)}\right)\tau \\
&+ \frac{2\tilde{\kappa}\tilde{\theta}}{\sigma_v^2} \times ln\left(\frac{1-\tilde{q}(u)}{1-\tilde{q}(u)e^{-\tilde{d}(u_1)\tau}}\right) \\
&- \frac{2\sigma_v^2\tilde{\mu}_\xi\tilde{l}_0}{\left(\tilde{\mu}_\xi\left(\tilde{\kappa} + \tilde{d}(u_1)\right) - \sigma_v^2\right)\left(\sigma_v^2 - \tilde{\mu}_\xi\left(\tilde{\kappa} - \tilde{d}(u_1)\right)\right)} \\
&\times ln\left(\frac{\sigma_v^2 - \tilde{\mu}_\xi\left(\tilde{\kappa} - \tilde{d}(u_1)\right) + \left(\tilde{\mu}_\xi\left(\tilde{\kappa} + \tilde{d}(u_1)\right) - \sigma_v^2\right)\tilde{q}(u)}{\sigma_v^2 - \tilde{\mu}_\xi\left(\tilde{\kappa} - \tilde{d}(u_1)\right) + \left(\tilde{\mu}_\xi\left(\tilde{\kappa} + \tilde{d}(u_1)\right) - \sigma_v^2\right)\tilde{q}(u)e^{-\tilde{d}(u_1)\tau}}\right).
\end{aligned} \tag{4.33}$$

CHAPTER 5

VIX-squared futures premium

In this chapter, we will study the risk premium. The VIX, also known by volatility index, is an indicator that represents the market expectation for future volatility. The VIX-squared measure the future expected variance.

Martin (2017, Equation (25)) Martin 2017 shows that the time-T VIX-squared for a time-to-maturity of δ years is equal to

$$VIX_T^2 = \frac{2}{\delta} L_T^* \left(\frac{\frac{P_{T+\delta}}{P_T}}{P^{-1}(T,T+\delta)} \right),$$

where $L_T^*(X) = ln\ \mathbb{E}_T^*(X) - \mathbb{E}_T^*\ ln(X)$, P_T is the time-T SPX price, $P(T,T+\delta)$ is the time-T present value of a US\$ 1 payable at time $T+\delta$ and its is assumed that the underlying asset does not pay dividends in the time interval $[T,T+\delta]$.

Then,

$$\begin{aligned}
VIX_T^2 &= \frac{2}{\delta}\left[ln\ \mathbb{E}_T^* \left(\frac{\frac{P_{T+\delta}}{P_T}}{P^{-1}(T,T+\delta)} \right) - \mathbb{E}_T^*\ ln \left(\frac{\frac{P_{T+\delta}}{P_T}}{P^{-1}(T,T+\delta)} \right) \right] \\
&= \frac{2}{\delta}\left[ln \left(\frac{1}{P^{-1}(T,T+\delta)} \mathbb{E}_T^* \left(\frac{P_{T+\delta}}{P_T} \right) \right) - \mathbb{E}_T^*\ ln \left(\frac{P_{T+\delta}}{P_T P^{-1}(T,T+\delta)} \right) \right] \\
&= \frac{2}{\delta}\left[ln \left(\frac{\mathbb{E}_T^*\left(P_{T+\delta}\right)}{P_T P^{-1}(T,T+\delta)} \right) - \mathbb{E}_T^*\ ln \left(\frac{P_{T+\delta}}{P_T P^{-1}(T,T+\delta)} \right) \right] \\
&= -\frac{2}{\delta}\left[\mathbb{E}_T^*\ ln \left(\frac{P_{T+\delta}}{P_T P^{-1}(T,T+\delta)} \right) \right] \\
&= -\frac{2}{\delta}\left[\mathbb{E}_T^*\left[ln\left(P_{T+\delta}\right)\right] - ln\left(P_T\right) - ln\left(P^{-1}(T,T+\delta)\right) \right] \\
&= -\frac{2}{\delta}\left[\mathbb{E}_T^*\left[ln\left(P_{T+\delta}\right)\right] - ln\left(P_T\right) + ln\left(P(T,T+\delta)\right) \right], \qquad (5.1)
\end{aligned}$$

because the absence of dividends implies that $\mathbb{E}_T^*\left(P_{T+\delta}\right) = P_T P^{-1}(T,T+\delta)$.

5.1. Cumulant generating function

The expectation 'contained' on the right-hand side of equation (5.1) will be computed through the cumulant generating function of $lnP_{T+\delta}$.

DEFINITION 5.1. *The cumulant generating function of $lnP_{T+\delta}$ is*

$$\phi^*(z;T,T+\delta) = ln\ \mathbb{E}_T^*\left[e^{z\times lnP_{T+\delta}}\right]. \tag{5.2}$$

Now, using equation (3.46)

$$\begin{aligned}
\phi^*(z;T,T+\delta) &= ln\ \mathbb{E}_T^*\left[exp\left(z\times\left(lnP_T - \lambda_x'\cdot X_T + (r_f+\lambda_0)\,\delta + \lambda_x'\cdot X_{T+\delta}\right)\right)\right] \\
&= ln\ \left[exp\left(z\times\left(lnP_T - \lambda_x'\cdot X_T + (r_f+\lambda_0)\,\delta\right)\right)\times\mathbb{E}_T^*\left[e^{z\cdot\lambda_x'\cdot X_{T+\delta}}\right]\right] \\
&= z\times\left(lnP_T - \lambda_x'\cdot X_T + (r_f+\lambda_0)\,\delta\right) + ln\ \mathbb{E}_T^*\left[e^{z\cdot\lambda_x'\cdot X_{T+\delta}}\right],
\end{aligned}$$

and adding the information of the equation (4.20), we have

$$\begin{aligned}
&\phi^*(z;T,T+\delta) \\
&= z\times\left(lnP_T - \lambda_x'\cdot X_T + (r_f+\lambda_0)\,\delta\right) \\
&\quad + ln\left(\exp\left[\tilde{\alpha}(z\cdot\lambda_x';T,T+\delta) + \tilde{\beta}'(z\cdot\lambda_x';T,T+\delta)\cdot X_T\right]\right) \\
&= z\left(lnP_T - \lambda_x'\cdot X_T + (r_f+\lambda_0)\,\delta\right) + \tilde{\alpha}(z\cdot\lambda_x';T,T+\delta) + \tilde{\beta}'(z\cdot\lambda_x';T,T+\delta)\cdot X_T \\
&= z\left(lnP_T - ln\ x_T - \lambda_\sigma\sigma_t^2 + (r_f+\lambda_0)\,\delta\right) \\
&\quad + \tilde{\alpha}(z\cdot\lambda_x';T,T+\delta) + \tilde{\beta}_1(z\cdot\lambda_x';T,T+\delta)ln\ x_T + \tilde{\beta}_2(z\cdot\lambda_x';T,T+\delta)\sigma_T^2 \\
&= z\left(lnP_T - ln\ x_T - \lambda_\sigma\sigma_T^2 + (r_f+\lambda_0)\,\delta\right) \\
&\quad + \tilde{\alpha}(z\cdot\lambda_x';T,T+\delta) + z\times ln\ x_T + \tilde{\beta}_2(z\cdot\lambda_x';T,T+\delta)\sigma_T^2 \\
&= z\left(lnP_T + (r_f+\lambda_0)\,\delta\right) + \tilde{\alpha}(z\cdot\lambda_x';T,T+\delta) + \left[\tilde{\beta}_2(z\cdot\lambda_x';T,T+\delta) - z\lambda_\sigma\right]\sigma_T^2. \qquad (5.3)
\end{aligned}$$

If we differentiate $\phi^*(z;T,T+\delta)$ of equation (5.2) in order to z and then take $z=0$, then

$$\frac{\partial}{\partial z}\phi^*(z;T,T+\delta)\mid_{z=0} = \frac{\mathbb{E}_T^*\left[lnP_{T+\delta}e^{zlnP_{T+\delta}}\right]}{\mathbb{E}_T^*\left[e^{zlnP_{T+\delta}}\right]}\mid_{z=0}$$

$$= \mathbb{E}_T^*\left[lnP_{T+\delta}\right]. \tag{5.4}$$

Using equations (5.3) and (5.4):

$$\mathbb{E}_T^*\left[lnP_{T+\delta}\right] = \frac{\partial}{\partial z}\phi^*(z;T,T+\delta)\mid_{z=0}$$

$$= \frac{\partial}{\partial z}\Big[z\left(lnP_T + (r_f+\lambda_0)\,\delta\right) + \tilde{\alpha}(z\cdot\lambda_x';T,T+\delta) + \left[\tilde{\beta}_2(z\cdot\lambda_x';T,T+\delta) - z\lambda_\sigma\right]\sigma_T^2\Big]$$

$$= lnP_T + (r_f+\lambda_0)\,\delta + \frac{\partial}{\partial z}\tilde{\alpha}(z\cdot\lambda_x';T,T+\delta)\mid_{z=0} + \left[\frac{\partial}{\partial z}\tilde{\beta}_2(z\cdot\lambda_x';T,T+\delta)\mid_{z=0} - \lambda_\sigma\right]\sigma_T^2. \tag{5.5}$$

Combining equation (5.1) with equation (5.5),

$$VIX_T^2 = -\frac{2}{\delta}\left[\mathbb{E}_T^*\left[lnP_{T+\delta}\right] - lnP_T + lnP\left(T,T+\delta\right)\right]$$

$$= -\frac{2}{\delta}\left[lnP_T + (r_f+\lambda_0)\,\delta + \frac{\partial\tilde{\alpha}}{\partial z}(z\cdot\lambda_x';T,T+\delta)\mid_{z=0} + \left[\frac{\partial\tilde{\beta}_2}{\partial z}(z\lambda_x';T,T+\delta)\mid_{z=0} - \lambda_\sigma\right]\sigma_T^2 - lnP_T + lnP\left(T,T+\delta\right)\right]$$

$$= -\frac{2}{\delta}\left[(r_f+\lambda_0)\,\delta + \frac{\partial\tilde{\alpha}}{\partial z}(z\lambda_x';T,T+\delta)\mid_{z=0} + \left[\frac{\partial\tilde{\beta}_2}{\partial z}(z\lambda_x';T,T+\delta)\mid_{z=0} - \lambda_\sigma\right]\sigma_T^2 + lnP\left(T,T+\delta\right)\right]. \tag{5.6}$$

To simplify the notation, let's denote

$$\frac{\partial \tilde{\alpha}}{\partial z}(z \cdot \lambda_x'; T, T+\delta) \mid_{z=0} = \frac{\partial \tilde{\alpha}}{\partial z} \mid_{z=0} \tag{5.7}$$

and

$$\frac{\partial \tilde{\beta}_2}{\partial z}(z \cdot \lambda_x'; T, T+\delta) \mid_{z=0} = \frac{\partial \tilde{\beta}_2}{\partial z} \mid_{z=0}, \tag{5.8}$$

yielding

$$VIX_T^2 = -\frac{2}{\delta}\left[(r_f + \lambda_0)\,\delta + \frac{\partial \tilde{\alpha}}{\partial z} \mid_{z=0} + \left[\frac{\partial \tilde{\beta}_2}{\partial z} \mid_{z=0} -\lambda_\sigma\right]\sigma_T^2 + lnP\,(T, T+\delta)\right]. \tag{5.9}$$

5.2. The VIX-squared risk premium

To calculate the risk premium, we take the difference between the expected value under the physical measure and the expected value under the risk-neutral measure:

$$\begin{aligned}
&\mathbb{E}_T\left[VIX_T^2\right] - \mathbb{E}_T^*\left[VIX_T^2\right] \\
&= \mathbb{E}_T\left[-\frac{2}{\delta}\left[(r_f + \lambda_0)\,\delta + \frac{\partial \tilde{\alpha}}{\partial z} \mid_{z=0} + \left(\frac{\partial \tilde{\beta}_2}{\partial z} \mid_{z=0} -\lambda_\sigma\right)\sigma_T^2 + lnP\,(T, T+\delta)\right]\right] \\
&\quad - \mathbb{E}_T^*\left[-\frac{2}{\delta}\left[(r_f + \lambda_0)\,\delta + \frac{\partial \tilde{\alpha}}{\partial z} \mid_{z=0} + \left(\frac{\partial \tilde{\beta}_2}{\partial z} \mid_{z=0} -\lambda_\sigma\right)\sigma_T^2 + lnP\,(T, T+\delta)\right]\right] \\
&= -\frac{2}{\delta}\left[(r_f + \lambda_0)\,\delta + \frac{\partial \tilde{\alpha}}{\partial z} \mid_{z=0} + \left(\frac{\partial \tilde{\beta}_2}{\partial z} \mid_{z=0} -\lambda_\sigma\right)\mathbb{E}_T\left[\sigma_T^2\right] + lnP\,(T, T+\delta)\right] \\
&\quad + \frac{2}{\delta}\left[(r_f + \lambda_0)\,\delta + \frac{\partial \tilde{\alpha}}{\partial z} \mid_{z=0} + \left(\frac{\partial \tilde{\beta}_2}{\partial z} \mid_{z=0} -\lambda_\sigma\right)\mathbb{E}_T^*\left[\sigma_T^2\right] + lnP\,(T, T+\delta)\right] \\
&= -\frac{2}{\delta}\left(\frac{\partial \tilde{\beta}_2}{\partial z} \mid_{z=0} -\lambda_\sigma\right)\left(\mathbb{E}_T\left[\sigma_T^2\right] - \mathbb{E}_T^*\left[\sigma_T^2\right]\right).
\end{aligned} \tag{5.10}$$

We know the sign of the last term, $\mathbb{E}_T\left[\sigma_T^2\right] - \mathbb{E}_T^*\left[\sigma_T^2\right]$, because Eraker and Wu (2017, Appendix A7) Eraker, Wu 2017 shows that $\mathbb{E}_T\left[\sigma_T^2\right] < \mathbb{E}_T^*\left[\sigma_T^2\right]$.

All that remains is to evaluate the signal of the function $\frac{\partial\tilde{\beta}_2}{\partial z}(z\cdot\lambda_x;T,T+\delta)\mid_{z=0} - \lambda_\sigma$.

Remember that $z\cdot\lambda_x = \begin{bmatrix} z \\ z\lambda_\sigma \end{bmatrix}$ and $\tilde{\beta}_2(z\cdot\lambda_x;T,T+\delta) = \frac{\tilde{\kappa}-\tilde{d}(z)-\left(\tilde{\kappa}+\tilde{d}(z)\right)\tilde{q}(z\cdot\lambda_x)e^{-\tilde{d}(z)\delta}}{\sigma_v^2\left(1-\tilde{q}(z\cdot\lambda_x)e^{-\tilde{d}(z)\delta}\right)}$.

Let's simplify the expression to make the derivative more easily to do:

$$\begin{aligned}\tilde{\beta}_2(z\cdot\lambda_x;T,T+\delta) &= \frac{-2\tilde{d}(z)+\tilde{\kappa}+\tilde{d}(z)-\left(\tilde{\kappa}+\tilde{d}(z)\right)\tilde{q}(z\cdot\lambda_x)e^{-\tilde{d}(z)\delta}}{\sigma_v^2\left(1-\tilde{q}(z\cdot\lambda_x)e^{-\tilde{d}(z)\delta}\right)} \\ &= \frac{\tilde{\kappa}+\tilde{d}(z)}{\sigma_v^2} - \frac{2\tilde{d}(z)}{\sigma_v^2\left(1-\tilde{q}(z\cdot\lambda_x)e^{-\tilde{d}(z)\delta}\right)}. \end{aligned} \tag{5.11}$$

To calculate the derivative of this function, we need to do some auxiliar calculations and to make this more simple to understand, we are doing simplifications in notation that are used in resolution:

$$\begin{aligned}\tilde{d}(0) &= \tilde{d}(z)\mid_{z=0} \\ &= \sqrt{\tilde{\kappa}^2+\sigma_v^2\left(2\gamma+1-0\right)0} \\ &= \sqrt{\tilde{\kappa}^2} = \tilde{\kappa}\ (\text{if } \tilde{\kappa}>0); \end{aligned} \tag{5.12}$$

$$\begin{aligned}\tilde{d}'(0) &= \frac{\partial\tilde{d}}{\partial z}(z)\mid_{z=0} \\ &= \frac{1}{2}\sigma_v^2\left(2\gamma+1-2z\right)\frac{1}{\tilde{d}(z)}\mid_{z=0} \\ &= \frac{1}{2}\sigma_v^2\left(2\gamma+1-0\right)\frac{1}{\tilde{d}(0)} \\ &= \frac{\sigma_v^2\left(2\gamma+1\right)}{2\tilde{\kappa}}; \end{aligned} \tag{5.13}$$

$$
\begin{aligned}
\tilde{q}(0) &= \tilde{q}(z \cdot \lambda_x) \mid_{z=0} \\
&= \frac{\tilde{\kappa} - \tilde{d}(0) - 0\sigma_v^2}{\tilde{\kappa} + \tilde{d}(0) - 0\sigma_v^2} \\
&= \frac{\tilde{\kappa} - \tilde{\kappa}}{\tilde{\kappa} + \tilde{\kappa}} \\
&= \frac{0}{2\tilde{\kappa}} = 0
\end{aligned}
\tag{5.14}
$$

and

$$
\begin{aligned}
\tilde{q}'(0) &= \frac{\partial \tilde{q}}{\partial z}(z \cdot \lambda_x) \mid_{z=0} \\
&= \frac{\left[-\tilde{d}'(z) - \lambda_\sigma \sigma_v^2\right]\left[\tilde{\kappa} + \tilde{d}(z) - z\lambda_\sigma \sigma_v^2\right] - \left[\tilde{\kappa} - \tilde{d}(z) - z\lambda_\sigma \sigma_v^2\right]\left[\tilde{d}'(z) - \lambda_\sigma \sigma_v^2\right]}{\left[\tilde{\kappa} + \tilde{d}(z) - z\lambda_\sigma \sigma_v^2\right]^2} \mid_{z=0} \\
&= \frac{\left[-\tilde{d}'(0) - \lambda_\sigma \sigma_v^2\right]\left[\tilde{\kappa} + \tilde{d}(0)\right] - \left[\tilde{\kappa} - \tilde{d}(0)\right]\left[\tilde{d}'(0) - \lambda_\sigma \sigma_v^2\right]}{\left[\tilde{\kappa} + \tilde{d}(0)\right]^2} \\
&= -\frac{\left[\frac{\sigma_v^2(2\gamma+1)}{2\tilde{\kappa}} + \lambda_\sigma \sigma_v^2\right] 2\tilde{\kappa}}{4\tilde{\kappa}^2} \\
&= -\frac{\sigma_v^2}{4\tilde{\kappa}^2}\left(2\gamma + 1 + 2\tilde{\kappa}\lambda_\sigma\right).
\end{aligned}
\tag{5.15}
$$

Back to the equation (5.12), doing the differentiation of $\tilde{\beta}_2(z \cdot \lambda_x; T, T+\delta)$ and evaluating with $z = 0$,

$$
\begin{aligned}
&\frac{\partial \tilde{\beta}_2}{\partial z}(z \cdot \lambda_x; T, T+\delta) \mid_{z=0} \\
&= \frac{\tilde{d}'(0)}{\sigma_v^2} - \frac{1}{\sigma_v^2} \frac{2\tilde{d}'(0)\left[1 - \tilde{q}(0)e^{-\tilde{d}(0)\delta}\right] - 2\tilde{d}(0)\left[-\tilde{q}'(0)e^{-\tilde{d}(0)\delta} + \tilde{q}(0)\tilde{d}'(0)\delta e^{-\tilde{d}(0)\delta}\right]}{\left[1 - \tilde{q}(0)e^{-\tilde{d}(0)\delta}\right]^2}.
\end{aligned}
\tag{5.16}
$$

Replacing what we known:

$$\frac{\partial\tilde{\beta}_2}{\partial z}(z\cdot\lambda_x;T,T+\delta)\mid_{z=0}$$

$$=\frac{2\gamma+1}{2\tilde{\kappa}}-\frac{1}{\sigma_v^2}\left(2\frac{\sigma_v^2\left(2\gamma+1\right)}{2\tilde{\kappa}}+2\tilde{\kappa}\left(-\frac{\sigma_v^2}{4\tilde{\kappa}^2}\left(2\gamma+1+2\tilde{\kappa}\lambda_\sigma\right)e^{-\tilde{\kappa}\delta}\right)\right)$$

$$=\frac{2\gamma+1}{2\tilde{\kappa}}-\frac{1}{2\tilde{\kappa}}\left(2\left(2\gamma+1\right)-\left(2\gamma+1+2\tilde{\kappa}\lambda_\sigma\right)e^{-\tilde{\kappa}\delta}\right)$$

$$=\ -\frac{1}{2\tilde{\kappa}}\left(2\gamma+1-\left(2\gamma+1+2\tilde{\kappa}\lambda_\sigma\right)e^{-\tilde{\kappa}\delta}\right)$$

$$=\ -\frac{1}{2\tilde{\kappa}}\left(2\gamma+1\right)\left(1-e^{-\tilde{\kappa}\delta}\right)+\lambda_\sigma e^{-\tilde{\kappa}\delta}.$$

Finally,

$$\frac{\partial\tilde{\beta}_2}{\partial z}(z\cdot\lambda_x;T,T+\delta)\mid_{z=0}-\lambda_\sigma$$

$$=-\frac{2\gamma+1}{2\tilde{\kappa}}\left(1-e^{-\tilde{\kappa}\delta}\right)+\lambda_\sigma e^{-\tilde{\kappa}\delta}-\lambda_\sigma$$

$$=-\frac{2\gamma+1}{2\tilde{\kappa}}\left(1-e^{-\tilde{\kappa}\delta}\right)-\lambda_\sigma\left(1-e^{-\tilde{\kappa}\delta}\right)$$

$$=-\left(\frac{2\gamma+1}{2\tilde{\kappa}}+\lambda_\sigma\right)\left(1-e^{-\tilde{\kappa}\delta}\right)$$

$$=-\frac{1}{2\tilde{\kappa}}\left(2\gamma+1+2\tilde{\kappa}\lambda_\sigma\right)\left(1-e^{-\tilde{\kappa}\delta}\right). \tag{5.17}$$

$1-e^{-\tilde{\kappa}\delta}>0$, if $\tilde{\kappa}>0$, but we can't evaluate the 'global' sign yet, because we don't know the sign of the second term.

To achieve this, we will return to the equation (3.51) in order to determine the sign of $2\gamma+1+2\tilde{\kappa}\lambda_\sigma$,

$$\lambda_\sigma = \frac{d(-\gamma) - d(1-\gamma)}{\sigma_v^2}$$

$$= \frac{\sqrt{\kappa^2 + \sigma_v^2 (-\gamma)(1+\gamma)} - \sqrt{\kappa^2 + \sigma_v^2 (1-\gamma)(1-(1-\gamma))}}{\sigma_v^2}$$

$$= \frac{\sqrt{\kappa^2 - \sigma_v^2 (\gamma^2+\gamma)} - \sqrt{\kappa^2 - \sigma_v^2 (\gamma^2-\gamma)}}{\sigma_v^2} < 0 \text{ (because } \gamma > 0) \tag{5.18}$$

Now, we have to change variable $\tilde{\kappa}$ by κ and for this, we use equation (4.15):

$$\tilde{\kappa} = \kappa + \eta\sigma_v^2$$

$$= \kappa - \beta_2(u_{-\gamma}; +\infty)\sigma_v^2$$

$$= \kappa - \frac{\kappa - d(-\gamma)}{\sigma_v^2}\sigma_v^2 = \kappa - \kappa + d(-\gamma)$$

$$= d(-\gamma) = \sqrt{\kappa^2 - \sigma_v^2 (\gamma^2+\gamma)} \; (> 0). \tag{5.19}$$

Then, using equation (5.18) and equation (5.19)

$$
\begin{aligned}
& 1+2\gamma+2\tilde{\kappa}\lambda_\sigma \\
&= 1+2\gamma+2\sqrt{\kappa^2-\sigma_v^2\left(\gamma^2+\gamma\right)}\left(\frac{\sqrt{\kappa^2-\sigma_v^2\left(\gamma^2+\gamma\right)}-\sqrt{\kappa^2-\sigma_v^2\left(\gamma^2-\gamma\right)}}{\sigma_v^2}\right) \\
&= 1+2\gamma+2\frac{\kappa^2-\sigma_v^2\left(\gamma^2+\gamma\right)}{\sigma_v^2}-2\frac{\sqrt{\kappa^2-\sigma_v^2\left(\gamma^2+\gamma\right)}\sqrt{\kappa^2-\sigma_v^2\left(\gamma^2-\gamma\right)}}{\sigma_v^2} \\
&= 1+2\gamma+2\frac{\kappa^2-\sigma_v^2\gamma^2-\sigma_v^2\gamma}{\sigma_v^2} \\
&\quad -2\frac{\sqrt{\kappa^4-\kappa^2\sigma_v^2\left(\gamma^2-\gamma\right)-\kappa^2\sigma_v^2\left(\gamma^2+\gamma\right)+\sigma_v^4\left(\gamma^2+\gamma\right)\left(\gamma^2-\gamma\right)}}{\sigma_v^2} \\
&= 1+2\frac{\kappa^2-\sigma_v^2\gamma^2}{\sigma_v^2}-2\frac{\sqrt{\kappa^4-\kappa^2\sigma_v^2\gamma^2-\kappa^2\sigma_v^2\gamma^2+\sigma_v^4\left(\gamma^4-\gamma^2\right)}}{\sigma_v^2} \\
&= 1+2\frac{\kappa^2-\sigma_v^2\gamma^2}{\sigma_v^2}-2\frac{\sqrt{\left(\kappa^2\right)^2-2\kappa^2\sigma_v^2\gamma^2+\left(\sigma_v^2\gamma^2\right)^2-\left(\sigma_v^2\gamma\right)^2}}{\sigma_v^2} \\
&= 1+2\frac{\kappa^2-\sigma_v^2\gamma^2}{\sigma_v^2}-2\frac{\sqrt{\left(\kappa^2-\sigma_v^2\gamma^2\right)^2-\left(\sigma_v^2\gamma\right)^2}}{\sigma_v^2}. \qquad (5.20)
\end{aligned}
$$

Corollary 5.1.

$1+2\gamma+2\tilde{\kappa}\lambda_\sigma$ *is positive.*

Proof.

Since

$$\sqrt{\left(\kappa^2-\sigma_v^2\gamma^2\right)^2-\left(\sigma_v^2\gamma\right)^2}<\sqrt{\left(\kappa^2-\sigma_v^2\gamma^2\right)^2}=\kappa^2-\sigma_v^2\gamma^2,$$

then

$$-\sqrt{\left(\kappa^2-\sigma_v^2\gamma^2\right)^2-\left(\sigma_v^2\gamma\right)^2}>-\sqrt{\left(\kappa^2-\sigma_v^2\gamma^2\right)^2}=-\left(\kappa^2-\sigma_v^2\gamma^2\right).$$

Therefore,

$$\kappa^2 - \sigma_v^2\gamma^2 - \sqrt{(\kappa^2 - \sigma_v^2\gamma^2)^2 - (\sigma_v^2\gamma)^2} > 0$$

Multiplying both sides by $\frac{2}{\sigma_v^2}$:

$$2\frac{\kappa^2 - \sigma_v^2\gamma^2}{\sigma_v^2} - 2\frac{\sqrt{(\kappa^2 - \sigma_v^2\gamma^2)^2 - (\sigma_v^2\gamma)^2}}{\sigma_v^2} > 0$$

Adding 1 to both sides,

$$1 + 2\frac{\kappa^2 - \sigma_v^2\gamma^2}{\sigma_v^2} - 2\frac{\sqrt{(\kappa^2 - \sigma_v^2\gamma^2)^2 - (\sigma_v^2\gamma)^2}}{\sigma_v^2} > 1 > 0.$$

The left-hand side is equal to equation (5.20), which is equal to $1 + 2\gamma + 2\tilde{\kappa}\lambda_\sigma$. □

Now, we have information to replace in into equation (5.17), and conclude that

$$\frac{\partial\tilde{\beta}_2}{\partial z}(z \cdot \lambda_x; T, T+\delta) \mid_{z=0} -\lambda_\sigma < 0. \tag{5.21}$$

Finally, back to the equation (5.10) and using all this, we've enough to conclude that long positions in VIX-squared futures must have negative returns.

CHAPTER 6

Conclusions

In this thesis, we explore the two-factor model proposed by Eraker and Wu (2017) in various financial contexts. Throughout in this research our main goal was to examine in detail the nuances of the model and understand the relevant implications to asset valuation and risk management.

We start our research by presenting the model and outlining your configuration in the physical measure domain. A detailed analysis of the model structure and the underlying stochastic processes was crucial to our comprehension. The representation in matricial notation fit well with our objectives, yielding a clearer and more concise analysis.

In the course of the study, we turn our attention to the risk-neutral measure, introducing the concept of stochastic discount factor. Again, we carefully detail the specification of the model under this measure.

The apogee of our research was the in-depth analysis of the premium associated to the futures contracts on the VIX (-squared) index, a relevant primordial concept in financial market volatility. Through the equations and results obtained along this study, we clarify the underlying fundamentals to this premium, thus demonstrating that such contracts have negative expected returns. That discovery has significant implications to risk management and hedging strategies, since it evidences the willingness of investors, that have risk aversion, in paying that premium to protect their portfolios during high volatility periods.

Conclusively, this thesis provides a deeper understanding of market fear, as the search for protection mechanisms reflects the inherent risk aversion on the part of investors. The evidence of this expected negative returns in specific contracts suggests that, over time, the investors are willing to pay in order to avoid that unexpected market volatility.

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