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M | G | ∞ Queue Heavy-Traffic Situation Busy Period Lenght Distribution (Power and Pareto Service Distributions)

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M | G | ∞ QUEUE HEAVY-TRAFFIC SITUATION BUSY PERIOD LENGTH DISTRIBUTION (POWER AND PARETO SERVICE DISTRIBUTIONS)

DISTRIBUIÇÃO DO COMPRIMENTO DO PERÍODO DE OCUPAÇÃO DA FILA DE ESPERA M | G | ∞ EM SITUAÇÃO DE "HEAVY-TRAFFIC" (DISTRIBUIÇÕES DE SERVIÇO POTÊNCIA E PARETO)

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ABSTRACT:

- (Ferreira e Ramalhoto, 1994) showed that for service distributions such that $\lim_{\alpha\to\infty} G(t) = 0$ ($G(\cdot)$ is the d.f. and α is the mean, for α and ρ (traffic intensity) great enough the M|G|∞ queue busy period length distribution is approximately exponential.

  In this work we show that for service distributions for which this condition is not necessarily true, it is possible to have in some situations an approximately exponential behaviour for the M|G|∞ queue busy period length.

KEY-WORDS:

- *M|G|∞, busy period, power distribution, Pareto distribution.*

RESUMO:

- (Ferreira e Ramalhoto, 1994) mostraram que para distribuições de serviço tais que $\lim_{\alpha\to\infty} G(t) = 0$ ($G(\cdot)$ é a função de distribuição e α a média), para α e ρ (intensidade de tráfego) suficientemente grandes a distribuição do comprimento do período de ocupação da fila de espera M|G|∞ é aproximadamente exponencial.

  Neste trabalho mostramos que para distribuições de serviço para as quais aquela condição não se verifica necessariamente, é possível em certas situações ter um comportamento aproximadamente exponencial para o comprimento do período de ocupação da fila de espera M|G|∞.

PALAVRAS-CHAVE:

- *M|G|∞, período de ocupação, distribuição potência, distribuição de Pareto.*

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1. INTRODUCTION

A M|G|∞ queue busy period is thought here as being a period that begins when a customer arrives at the system being it empty, ends when a customer leaves the system letting it empty and in it there is, always, at least one customer in the system.

So be a M|G|∞ queue where λ is the arrival Poisson process rate, $G(\cdot)$ is the service d.f., $\alpha = \int_0^{\infty}[1-G(t)]dt$ is the mean service time, $\rho = \lambda\alpha$ is the traffic intensity and each customer, when it arrives, finds always an available server.

B is the busy period length and $\overline{B}(s)$ is its Laplace-Stieltjes transform.

(Takács, 1962) showed that

$$\overline{B}(s) = 1 + \lambda^{-1}\left(s - \frac{1}{\int_0^{\infty} e^{-st-\lambda\int_0^t[1-G(v)]dv}dt}\right) \quad (1.1).$$

The inversion of **(1.1)** is very hard to carry out. The exception is the service distributions collection (Ferreira, 1998)

$$G(t) = 1 - \frac{(1-e^{-\rho})(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda},\ t \geq 0,\ -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho}-1} \quad (1.2)$$

for which B is exponentially distributed at rate $e^{-\rho}(\lambda+\beta)$, with an atom at the origin whose value is $\frac{e^{-\rho}(\lambda+\beta)-\beta}{\lambda}$.

In (Ferreira e Ramalhoto, 1994) we showed that for service distributions such that $\lim_{\alpha\to\infty} G(t) = 0$ (as it happens, for instance, with the exponential), for α and ρ great enough B is approximately exponentially distributed. It is an important result because it gives emphasis to a B distribution insensibility situation related with that service distributions class in heavy-traffic conditions.

In this work we will study service distributions cases for which we have not necessarily $\lim_{\alpha\to\infty} G(t) = 0$, as it happens with power and Pareto ones, in order to identify situations for which B has a similar behaviour.

To achieve that goal we will use the busy period assimetry coefficient

$$\beta_1 = \frac{\left(E\left[B^3\right] - 3E[B]E\left[B^2\right] + 2E^3[B]\right)^2}{\left(E\left[B^2\right] - E^2[B]\right)^3} \tag{1.3}$$

and the Kurtosis one

$$\beta_2 = \frac{E\left[B^4\right] - 4E[B]E\left[B^3\right] + 6E^2[B]E\left[B^2\right] - 3E^4[B]}{\left(E\left[B^2\right] - E^2[B]\right)^2} \tag{1.4}$$

that in the exponential distribution case have the values 4 and 9 respectively (see, for instance, (Kendall and Stuart, 1979) and (Murteira, 1979)).

To compute $E\left[B^n\right]$ we have, according to (Ferreira e Ramalhoto, 1994),

$$E\left[B^n\right] = (-1)^{n+1}\left\{\frac{e^\rho}{\lambda} nC^{n-1} - e^\rho \sum_{p=1}^{n-1} \binom{n}{p} E\left[B^{n-p}\right] C^p\right\}, \quad n = 1,2,... \tag{1.5}$$

where

$$C^n = \int_0^\infty (-t)^n e^{-\lambda \int_0^t [1-G(v)]dv} \lambda(1 - G(t))dt, \quad n = 0,1,2,... \tag{1.6}$$

and calculated the integrals numerically with the aid of a computer.

2. POWER SERVICE DISTRIBUTION

If the service distribution is a power function with parameter C, $C > 0$

- $G(t) = \begin{cases} t^C, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

- $\alpha = \frac{C}{C+1}$ and $0 < \alpha < 1$.

So $\lim_{C\to\infty} G(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$ and $\lim_{C\to\infty} \alpha = 1$.

Being $B(t)$ the d.f. of B, from **(1.1)** we have, see (Stadje, 1985),

$$B(t) = 1 - \lambda^{-1} \sum_{n=1}^{\infty} \left[\frac{e^{-\lambda \int_0^t [1-G(v)]dv} \lambda(1-G(t))}{1-e^{-\rho}} \right]^{*n} \left(1-e^{-\rho}\right)^n \qquad (2.1)$$

where $*$ is the convolution operator.

For the service distribution with which we are working, since ρ and C are great enough,

$$\frac{e^{-\lambda \int_0^t [1-G(v)]dv} \lambda(1-G(t))}{1-e^{-\rho}} \cong \lambda e^{-\lambda t}, \quad 0 \le t \le 1.$$

Computing the Laplace-Stieltjes transform of **(2.1)** with this approximation we have

$$\int_0^1 e^{-st} \lambda e^{-\lambda t} dt = \lambda \int_0^1 e^{-(s+\lambda)t} dt = \lambda \left[-\frac{e^{-(s+\lambda)t}}{s+\lambda} \right]_0^1 = \lambda \frac{1-e^{-(s+\lambda)}}{s+\lambda}.$$

But, for λ great enough, we have approximately $\frac{\lambda}{\lambda+s}$. So, from **(2.1)** taking the Laplace-Stieltjes transform of $\frac{e^{-\lambda \int_0^t [1-G(v)]dv} \lambda(1-G(t))}{1-e^{-\rho}}$ as being $\frac{\lambda}{\lambda+s}$ we conclude that $B(t) \cong 1 - e^{-\lambda e^{-\rho} t}$, $t \ge 0$ that is:

- In a M|G|∞ queue where the service distribution is a power function, for α near 1, since α and ρ are great enough B is approximately exponential with mean $\frac{e^{\rho}}{\lambda}$.

For this system we computed the values of β_1 and β_2 given for **(1.3)** and **(1.4)**, respectively, for $\alpha = .25$, $.5$ and $.8$ making, in each case, ρ take values from $.5$ till 100.

We got the following results:

ρ	$\alpha = .25$		$\alpha = .5$		$\alpha = .8$	
	β_1	β_2	β_1	β_2	β_1	β_2
.5	3.0181197	9.5577742	1.5035507	5.9040102	3.8933428	9.3287992
1	4.4211164	9.1402097	2.7111584	7.4994861	3.9854257	9.0702715
1.5	5.3090021	10.433228	3.3711526	8.2784408	3.9749455	8.9969919
2	5.8206150	11.140255	3.7332541	8.6924656	3.9751952	8.9815770
2.5	6.0803833	11.489308	3.9322871	8.9173048	3.9809445	8.9828631
3	6.1786958	11.619970	4.0388433	9.0369125	3.9871351	8.9877124
6	5.7006232	11.020248	4.0969263	9.1024430	3.9996462	8.9996459
7	5.5034253	10.774653	4.0765395	9.0804332	3.9999342	8.9999341
8	5.3382992	10.570298	4.0596336	9.0623268	3.9999992	8.9999992
9	5.2037070	10.404722	4.0467687	9.0486486	4.0000086	9.0000086
10	5.0944599	10.271061	4.0372385	9.0385796	4.0000068	9.0000068
15	4.7702550	9.8790537	4.0152698	9.0156261	4.0000005	9.0000005
20	4.6102773	9.6888601	4.0082556	9.0083980	4.0000000	9.0000000
50	4.3045903	9.3338081	4.0012425	9.0012513	4.0000000	9.0000000
100	4.1715617	9.1842790	4.0003047	9.0003057	4.0000000	9.0000000

The analysis of the results shows a strong trend of β_1 and β_2, to 4 and 9, respectively, after $\rho = 10$. This trend is faster the greatest is the value of α.

3. PARETO SERVICE DISTRIBUTION

In this section we will see, mainly, examples. So let us consider a Pareto distribution such that

$$1 - G(t) = \begin{cases} 1, & t < k \\ \left(\frac{k}{t}\right)^3, & t \geq k \end{cases}, \quad k > 0 \qquad (3.1).$$

Then, $\alpha = \frac{3}{2}k$ (see, for instance, (Murteira, 1979)).

The values calculated for β_1 and β_2 given for **(1.3)** and **(1.4)**, respectively with $\lambda = 1$ and, so, $\rho = \alpha$ were

$\alpha = \rho$	β_1	β_2
.5	1028.5443	1373.4466
1	1474.7159	1969.0197
10	38.879220	54.896896
20	4.0048588	9.0049233
50	4.0000000	9.0000000
100	4.0000000	9.0000000

and show a strong trend from β_1 and β_2 to 4 and 9, respectively, after $\rho = 20$. This is natural because, in this case, the convergence of α to infinite makes k to have the same behaviour and, so, after **(3.1)** we have $\lim\limits_{\alpha \to \infty} G(t) = 0$

But, considering now a Pareto distribution, such that

$$1 - G(t) = \begin{cases} 1, & t < .4 \\ \left(\dfrac{.4}{t}\right)^{\theta}, & t \geq .4 \end{cases}, \quad \theta > 1 \qquad (3.2),$$

$\alpha = \dfrac{.4\theta}{\theta - 1}$ (see, still, (Murteira, 1979)) and the values got for β_1 and β_2 in the same conditions of the previous case are

$\alpha = \rho$	β_1	β_2
.5	10.993704	16.675733
1	6.8553306	12.010791
10	4.5112470	9.5724605
20	4.4832270	9.5397410
50	4.4669879	9.5208253
100	4.4616718	9.5146406

and do not go against the hypotesis of the existence of a trend from β_1 and β_2 to 4 and 9, respectively, although much slower than in the previous case. But, now, the convergence of α to infinite implies do convergence of θ to 1. So

$$\lim_{\alpha \to \infty} G(t) = \begin{cases} 0, & t < .4 \\ 1 - \frac{.4}{t}, & t \geq .4 \end{cases}$$ and we can not guarantee at all that for α great enough $1 - G(t) \cong 1$.

4. CONCLUSIONS

In the case of service distributions for which is possible to get conditions for which $G(t) \cong 0$, for these conditions and with the possible consideration of another ones it is possible to guarantee that B is approximately exponentially distributed.

So, if $G(t) \cong 0$ for α great enough it is sufficient to consider ρ great enough.

But, for instance, if the service distribution is a power function, as we have seen we can not guarantee such conditions. However, for α near 1 and λ and ρ great enough it is possible to guarantee that B is approximately exponentially distributed.

But in the case of Pareto distribution where we have not $G(t) \cong 0$ for α great enough, although we can not give identical guarantees to those of the power function service, the results got for β_1 and β_2 are not against that for ρ great enough B is approximately exponentially distributed.

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