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A M|G|∞ SYSTEMS COLLECTION WITH EXPONENTIAL BUSY PERIOD

UMA COLECÇÃO DE SISTEMAS M|G|∞ COM PERÍODO DE OCUPAÇÃO EXPONENCIAL

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ABSTRACT:

- This paper describes how studying the transient behaviour of the M|G|∞ system, starting at the beginning of a busy period, it is possible to determine a collection of service distributions for which the length of the busy period is exponential, with an atom at the origin. We determine also the distribution of the busy cycle length for these systems and the mean number of busy periods that begin in $[0,t]$.

KEY-WORDS:

- *M|G|∞, Busy Period, Busy Cycle, Ricatti's Equation, Exponential Distribution.*

RESUMO:

- Este trabalho descreve como o estudo do comportamento transeunte do sistema M|G|∞, sendo o instante inicial o do começo de um período de ocupação, permite determinar uma colecção de distribuições de serviço para as quais o comprimento do período de ocupação é exponencial, com uma concentração de probabilidade na origem. Determina-se também a distribuição do comprimento do ciclo de ocupação para estes sistemas e o número médio de períodos de ocupação que se iniciam em $[0,t]$.

PALAVRAS-CHAVE:

- *M|G|∞, Período de Ocupação, Ciclo de Ocupação, Equação de Ricatti, Distribuição Exponencial.*

1. INTRODUTION

The determination of the busy period length distribution is one of the most delicate subjects in queues studies. Here we present a service distributions collection for which the $M|G|\infty$ queue busy period has exponential distribution, with an atom at the origin.

In a $M|G|\infty$ queue the arrival process is Poisson, at rate λ, and every customer meets, upon its arrival, a server available and receives a service whose length is a positive random variable with distribution function $G(.)$. We will call $i(t)$ the integral between 0 and t of $1-G(v)$, being v the variable, and we have $i(\infty)=\alpha$, being α the mean service time. The traffic intensity, ρ, is given for $\rho=\lambda\alpha$.

A queue system busy period begins when a customer arrives at the system finding it empty, ends when a customer leaves it letting it empty, and during it there is always at least one customer in the queue.

So, in a queue system, there is a sequence of busy and idle periods. A busy period followed by an idle period is called a busy cycle.

Calling $p_{1'0}(t)$ the $M|G|\infty$ system emptiness probability at time t, being the time origin the beginning of a busy period, and $p_{00}(t)$ the same probability in the situation of initially empty system, we have

$$p_{1'0}(t)=G(t)p_{00}(t) \quad (1.1).$$

In fact, so that the system is empty at time t, the customer that arrived in the initial instant must have left, what happens with probability $G(t)$, and the servers that were unoccupied at time origin must go on unoccupied at time t, what happens with probability $p_{00}(t)$. The independence of these two facts justifies (1.1).

As $p_{00}(t)=\exp[-\lambda i(t)]$(Harrison and Lemoine, 1981), from (1.1), we have

$$p_{1'0}(t)=G(t)\exp[-\lambda i(t)] \quad (1.2).$$

2. DETERMINATION OF THE SERVICE DISTRIBUTIONS COLLECTION FOR WHICH THE BUSY PERIODS M|G|∞ QUEUE IS EXPONENTIALLY DISTRIBUTED

The M|G|∞ queue busy period length Laplace-Stieltjes transform, $\overline{B}(s)$, is given by

$$\overline{B}(s)=\frac{p_{1'0}(s)}{p_{00}(s)} \qquad (2.1)$$

where $p_{1'0}(s)$ and $p_{00}(s)$ are the Laplace-Stieltjes transforms of $p_{1'0}(t)$ and $p_{00}(t)$, respectively (Ferreira, 1995).

From (1.2) $\frac{d}{dt}p_{1'0}(t)=p_{00}(t)(1-G(t))\left(\frac{g(t)}{1-G(t)}-\lambda G(t)\right)$. Putting

$\frac{g(t)}{1-G(t)}-\lambda G(t)=\beta$, $t>0$ with β real, we get the differential equation

$$\frac{dG(t)}{dt}=-\lambda G^2(t)+(\lambda-\beta)G(t)+\beta \qquad (2.2)$$

that is a Ricatti equation (see, for instance, (Ferreira, 1994)). To solve it we have to put $G(t)=u(t)+\frac{1}{z(t)}$ where $u(t)$ is a solution of (2.2). It is easily seen that we can have $u(t)=1$ getting

$$\frac{dz}{dt}-(\lambda+\beta)z=\lambda \qquad (2.3).$$

The equation (2.3) is a first order linear differential equation with constant coefficients easily solvable. So we get

$$G(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda},\quad t\geq 0,\ -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \qquad (2.4)$$

that is the required distribution functions collection. In fact putting (2.4) at (2.1) we get

$$\overline{B}(s)=\frac{e^{-\rho}(\lambda+\beta)(\lambda+s)-s\beta}{\lambda\left(e^{-\rho}(\lambda+\beta)+s\right)} \tag{2.5}$$

whose inversion gives the p.d.f. for the M | G | ∞ queue busy period length

$$b(t)=\frac{e^{-\rho}(\lambda+\beta)-\beta}{\lambda}\delta(t)+\left(1-\frac{e^{-\rho}(\lambda+\beta)-\beta}{\lambda}\right)e^{-\rho}(\lambda+\beta)e^{-e^{-\rho}(\lambda+\beta)t},$$

$$t\geq 0,-\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \qquad (2.6),$$

being $\delta(.)$ the Dirac function.

Note that:

- $\beta=-\lambda$

Then

$$G(t)=1,\quad t\geq 0 \text{ and } b(t)=\delta(t) \tag{2.7}$$

(see (Ferreira , 1988)).

- $\beta=0$

Then

$$G(t)=\frac{e^{-\rho}}{e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\lambda t}},\quad t\geq 0 \tag{2.8}$$

and

$$b(t)=e^{-\rho}\delta(t)+\left(1-e^{-\rho}\right)e^{-\rho}\lambda e^{-e^{-\rho}\lambda t},\quad t\geq 0 \tag{2.9}$$

(see (Ferreira , 1991)).

- $\beta=\frac{\lambda}{e^{\rho}-1}$

Then

$$1-G(t)=\frac{1}{1-e^{-\rho}+e^{-\rho+\frac{\lambda}{1-e^{-\rho}}t}},\quad t\geq 0 \tag{2.10}$$

and

$$b(t)=\frac{\lambda}{e^{\rho}-1}e^{-\frac{\lambda}{e^{\rho}-1}t},\quad t\geq 0 \tag{2.11}$$

(see (Ferreira, 1995b)).

The distribution function corresponding to (2.6) is

$$B(t)=1-\frac{\lambda+\beta}{\lambda}\left(1-e^{-\rho}\right)e^{-e^{-\rho}(\lambda+\beta)t},\quad t\geq 0,\ -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \tag{2.12}$$

If $0\leq\beta\leq\frac{\lambda}{e^{\rho}-1}$, for α great enough so that $1-e^{-\rho}\cong 1$, $G(t)\cong 0$ and $B(t)\cong 1-e^{-e^{-\rho}\lambda t}$ (see (Ferreira e Ramalhoto, 1994)).

If $-\lambda<\beta<0$, for α great enough so that $1-e^{-\rho}\cong 1$, $G(t)\cong-\frac{\beta}{\lambda}$ and $B(t)\cong 1-\frac{\lambda+\beta}{\lambda}e^{-e^{-\rho}(\lambda+\beta)t}$. So, only for λ great enough we have $G(t)\cong 0$ and $B(t)\cong 1-e^{-e^{-\rho}\lambda t}$ (see (Ferreira, 1996)).

That is, the busy period, for these systems, shows exponential behaviour in "heavy-traffic" situations.

3. BUSY CYCLE

The M|G|∞ queue busy length Laplace-Stieltjes transform is given by

$$\overline{Z}(s)=\frac{\lambda}{\lambda+s}\overline{B}(s) \tag{3.1}$$

because the idle period length is exponential with parameter λ (as it happens with any queue systems with Poisson arrival process) and is independent of the busy period length (Takács, 1962).

Substituting (2.5) in (3.1) and inverting we get the $M|G|\infty$ queue busy cycle length p.d.f.

$$z(t)=\frac{e^{-\rho}\left(1-e^{-\rho}\right)(\lambda+\beta)^2}{\lambda-e^{-\rho}(\lambda+\beta)}e^{-e^{-\rho}(\lambda+\beta)t}-\frac{\lambda\beta}{\lambda-e^{-\rho}(\lambda+\beta)}e^{-\lambda t},$$

$$t\geq 0,\ -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \qquad (3.2).$$

So,

- $\beta=-\lambda$

$$z(t)=\lambda e^{-\lambda t},\quad t\geq 0, \qquad (3.3)$$

evidently,

- $\beta=0$

$$z(t)=\lambda e^{-\rho}e^{-\lambda e^{-\rho}t},\quad t\geq 0, \qquad (3.4)$$

(see (Ferreira , 1995a)).

- $\beta=\dfrac{\lambda}{e^{\rho}-1}$

$$z(t)=\frac{\lambda}{e^{\rho}-2}\left(e^{-\frac{\lambda}{e^{\rho}-1}}-e^{-\lambda t}\right),\ t\geq 0, \qquad (3.5).$$

4. MEAN NUMBER OF BUSY PERIODS THAT BEGIN IN $[0,t]$

Be the M|G|∞ queue with time origin at the beginning of a busy period. The instants $0,\ t_1,\ t_2,\ldots$ at which begin a busy period, are the arrival moments of a renewal process (Takács, 1962). This renewal process function is (Ferreira, 1995)

$$R(t)=e^{-\lambda\int_0^t[1-G(v)]dv}+\lambda\int_0^t e^{-\lambda\int_0^u[1-G(v)]dv}du \tag{4.1}$$

and gives the mean number of busy periods that begin in $[0,t]$.

Substituting (2.5) in (4.1) we get

$$R(t)=1+\lambda t,\quad \beta=-\lambda \tag{4.2}$$

according to (3.3), and

$$R(t)=e^{-\rho}(1+\lambda t)+\left(1-e^{-\rho}\right)\frac{\beta}{\lambda+\beta}e^{-(\lambda+\beta)t}+\left(1-e^{-\rho}\right)\frac{\lambda}{\lambda+\beta},$$

$$-\lambda<\beta\le\frac{\lambda}{e^{\rho}-1} \tag{4.3}.$$

So,

$-\beta=0$

$$R(t)=1+\lambda e^{-\rho}t \tag{4.4}$$

according to (3.4),

$$-\beta=\frac{\lambda}{e^{\rho}-1}$$

$$R(t)=e^{-\rho}+\left(1-e^{-\rho}\right)^2+\lambda e^{-\rho}t+e^{-\rho}\left(1-e^{-\rho}\right)e^{-\frac{\lambda}{1-e^{-\rho}}t} \tag{4.5}$$

(see, (Ferreira , 1995)).

5. CONCLUSIONS

Imposing $\frac{g(t)}{1-G(t)} - \lambda G(t)$ to be constant, we find a service distributions family for which the M|G|∞ queue busy period length has a distribution that is a mixture of an origin degenerated distribution and an exponential one.

Other distributions already determinated are members of this family for which the busy period length has a similar behaviour.

We have, so, a M|G|∞ systems collection for which everything is known about the busy period distribution and about the busy cycle distribution.

It happens also that $b(t)$ and $z(t)$, for these M|G|∞ systems collection, have quite simple structures that do not rise any problem as for the distribution function computation and as for the moments of any order.

Finally, for such an important parameter as the mean number of busy periods that begin in $[0,t]$, we get also quite simple expressions for these systems.

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