Filomat 35:8 (2021), 2693–2704
https://doi.org/10.2298/FIL2108693B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# On the Stability of a Quadratic Functional Equation Over Non-Archimedean Spaces

**Gastão Bettencourt[a], Sérgio Mendes[b,c]**

[a]*Universidade da Beira Interior, Centro de Matemática e Aplicações (CMA-UBI), Rua Marquês d'Ávila e Bolama,6200-001, Covilhã, Portugal*
[b]*ISCTE - Lisbon University Institute,Av. das Forças Armadas, 1649-026, Lisbon, Portugal*
[c]*Centro de Matemática e Aplicações (CMA-UBI), Rua Marquês d'Ávila e Bolama, 6200-001, Covilhã, Portugal*

**Abstract.** Let $G$ be an abelian group and suppose that $X$ is a non-Archimedean Banach space. We study Hyers-Ulam-Rassias stability for the functional equation of quadratic type

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x)$$

where $f: G \to X$ is a map.

## 1. Introduction

Groups are among the most studied objects in mathematics and a source of many problems and conjectures. An example is the stability problem for a group homomorphism, raised by Ulam [26] in 1940. It can be stated as follows. Suppose $f(x)$ satisfies only approximately the Cauchy functional equation $h(x+y)=h(x)+h(y)$. Does there exist a solution $\varphi$ of the Cauchy functional equation which approximates $f(x)$? To make the statement more general and precise, let $G_1$ be a group and let $G_2$ be a group endowed with a metric $d$. Given $\varepsilon > 0$, we ask if there exist a $\delta > 0$ such that, if a map $f : G_1 \to G_2$ verifies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$ then a homomorphism $\varphi : G_1 \to G_2$ exists, with $d(f(x), \varphi(x)) < \varepsilon$ for all $x \in G_1$.

The solution of Ulam's problem was given a year later by Hyers [7] for the case when $G_1 = E$ and $G_2 = F$ are Banach spaces. Let $f : E \to F$ be a map such that $t \in \mathbb{R} \to f(tx)$ is continuous in $\mathbb{R}$ for each fixed $x \in E$. Suppose there exists constants $\varepsilon > 0$ and $p \in [0, 1)$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x\|^p+\|y\|^p), \tag{1}$$

for all $x, y \in E$. Rassias [19] proved that there exists a unique linear map $T : E \to F$ such that

$$\|f(x)-T(x)\| \leq \frac{\varepsilon}{1-2^{p-1}}\|x\|^p,$$

2020 *Mathematics Subject Classification*. Primary 46S10, 39B52; Secondary 26E30, 12J25
*Keywords*. Hyers-Ulam stability, Fréchet functional equation, length function
Received: 08 July 2020; Revised: 21 September 2020; Accepted: 13 October 2020
Communicated by Hemen Dutta
Research supported by Fundação para a Ciência e Tecnologia through Centro de Matemática e Aplicações da Universidade da Beira Interior (CMA-UBI), project UIDB/MAT/00212/2020.
*Email addresses:* gastao@ubi.pt (Gastão Bettencourt), sergio.mendes@iscte-iul.pt (Sérgio Mendes)

for all $x \in E$. In [5], Găvruta generalized further Rassias' result by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ in (1) by a control function $\varphi(x, y)$, which guaranties the existence of the unique linear map $T : E \to F$.

Since then, many stability problems, concerning not only homomorphisms but rather general functional equations, have been formulated and solved by several mathematicians, see for instance the articles in [20] and the bibliography therein. Recently, the stability of rational functional equations has been studied by Senthil Kumar and Dutta [11, 12] in the non-Archimedean setting using direct and fixed points methods. And in [13], the same authors tackled the stability problem in the setting of fuzzy Banach spaces with an application to systems design. In the same vein, Senthil Kumar, Dutta and Sabarinathan [14] considered stability of rational functional equations over the hyperreals and applications to physics were presented. There are several recent papers focusing on the stability of functional equations and fractional differential equations, see for example [15, 21, 22].

Let $G_1$, $G_2$ be abelian groups. Let $f : G_1 \to G_2$ be a map. In this article we are concerned with the functional equation

$$f(x + y + z) + f(x) + f(y) + f(z) = f(x + y) + f(y + z) + f(z + x) \tag{2}$$

with $x, y, z \in G_1$. Another functional equation related with equation (2) is the following. A map $g : G_1 \to G_2$ is said to verify the quadratic functional equation provided that

$$g(x + y) + g(x - y) = 2g(x) + 2g(y) \tag{3}$$

holds for all $x, y \in G_1$, in which case $g$ is called quadratic. The name comes from the fact that $g(x) = cx^2$ is a solution of (3) (and also of (2)) when $G_1 = G_2 = \mathbb{R}$ or any commutative ring.

The functional equation (2) has received different names in the literature. In [24, p. 194], the author refers to it as the functional equation of Deeba. See also the discussion in [9, p. 368]. However in [25, p. 247] the author calls equation (2) Fréchet's functional equation. In any case, its origins seems to be rooted in inner product spaces. In fact, among the many characterizations of inner product spaces [17], Fréchet [4] proved in 1935 that a normed space $(X, \|\,.\,\|)$ is an inner product space if and only if $f(x) = \|x\|$ verifies equation (2). Later on, Jordan and von Neumann [8] gave a simpler characterization by showing that $(X, \|\,.\,\|)$ is an inner product space if and only if the map $g(x) = \|x\|^2$ is a solution of the quadratic equation (3). This is the classical parallelogram law.

An interesting fact is that any solution of (2) is necessarily quadratic, i.e. verifies the quadratic functional equation (3). This is a matter of simple calculation, taking first $x = y = z = 0$ to see that $f(0) = 0$ and then taking $z = -x$. A solution of functional equation (2) when $G_1$ is an abelian group and $G_2$ is a normed space over a field with characteristic different from 2 was given by Kannappan [9], who showed that $f$ has the form $f(x) = B(x, x) + A(x)$, where $B : G_1 \times G_1 \to G_2$ is a biadditive map and $A : G_1 \to G_2$ is an additive map.

The stability of the functional equations (2) and (3) was studied by Jung [23] when $G_1$ is a real normed space and $G_2$ is a real Banach space. In [10], Kim investigated the stability of equation (2) in the sense of Găvruta. Hyers-Ulam stability in the non-Archimedean setting was initiated with the work of Arriola and Beyer [3] on the Cauchy functional equation for $G_1 = \mathbb{Q}_p$ and $G_2 = \mathbb{R}$. In [16], Moslehian and Rassias studied the stability of the Cauchy functional equation and quadratic equation for $G_1$ respectively, an abelian semigroup and an abelian group, and $G_2$ a non-Archimedean Banach space. Another functional equation related to the characterization of inner product spaces was studied, from the point of view of stability, by Gordji et al [6] also in the non-Archimedean setting.

In this article we study Hyers-Ulam-Rassias stability of functional equation (2) for the case where $G_1 = G$ is an abelian group and $G_2 = X$ is a non-Archimedean Banach space. We will use the direct method introduced by Hyers, that is, an additive map $A : G \to X$ and a quadratic map $Q : G \to X$ will be explicitly constructed from certain solutions $f : G \to X$ of equation (2). Quite specific, we have:

$$A(x) = \lim_{n\to+\infty} \frac{f(2^n x)}{2^n} \quad \text{and} \quad Q(x) = \lim_{n\to+\infty} \frac{f(2^n x)}{4^n},$$

where the first map $f$ is odd and the latter is even. For the upper bound, following the original idea of Găvruta, we consider a certain control function which will be made explicit. The article is organized as follows. In section 2 we introduce some notation and review briefly non-Archimedean normed spaces. In section 3 we study Hyers-Ulam-Rassias stability of equation (2). The first case, in §3.1, deals with the stability under an approximate odd solution, see Theorem 3.1. Then, along the lines of [16], we establish a concrete realization of the control function $\psi : G^3 \to [0, +\infty)$ by considering a suitable real map $\rho : [0, +\infty) \to [0, +\infty)$. In §3.2, the stability of equation (2) under an approximate even solution is proved, with the additional condition that the function $f : G \to X$ vanishes at $x = 0$. This is the contents of Theorem 3.3. A concrete realization of the control function is obtained, similarly to the odd case. Section 4 is dedicated to establish the main result. We prove in Theorem 4.1 Hyers-Ulam-Rassias stability for the functional equation (2), that is, $C^2 f(x, y, z) = 0$, where $C^2 f(x, y, z)$ is the second Cauchy difference of the map $f : G \to X$. In the proof, we take advantage of the canonical decomposition of $f$ into an odd and even part, which allows us to apply the previous results of section 3.

## 2. Background

The real numbers verify the Archimedean property: given $a, b > 0$, there exist an integer $n$ such that $a < nb$. Hensel introduced the field of $p$-adics which does not have the Archimedean property. The topological fields which do not satisfy the Archimedean property are called non-Archimedean fields.

**Definition 2.1.** *A non-Archimedean field is a field $\mathbb{K}$ endowed with a function $|\cdot| : \mathbb{K} \to \mathbb{R}$, such that, for every $a, b \in \mathbb{K}$, verifies:*

(*i*) $|a| \geq 0$ *and* $|a| = 0$ *if and only if* $a = 0$;

(*ii*) $|ab| = |a||b|$;

(*iii*) $|a + b| \leq \max\{|a|, |b|\}$.

The function $|\cdot| : \mathbb{K} \to \mathbb{R}$ is called a non-Archimedean valuation over $\mathbb{K}$. Property (*iii*) is called the strong triangle inequality. From (*iii*) we conclude that, for every integer $n \in \mathbb{Z}$,

$$|n| = |1 + 1 + \cdots + 1| \leq \max\{|1|, \cdots, |1|\} = 1.$$

Through the article we always denote the non-Archimedean valuation of the ground field by $|\cdot|$.

**Definition 2.2.** *Let $X$ be a vector space over a non-Archimedean field $\mathbb{K}$ with a nontrivial valuation. A non-Archimedean norm over $X$ is a function $\|\cdot\| : X \to \mathbb{R}$ such that, for every $x, y \in X$, and $a \in \mathbb{K}$,*

(*i*) $\|x\| \geq 0$ *and* $\|x\| = 0$ *if and only if* $x = 0$;

(*ii*) $\|ax\| = |a|\|x\|$;

(*iii*) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$.

A space with a non-Archimedean norm $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

**Example 2.3.** *Let $p$ be a prime number. For every nonzero rational number $x$, there exists a unique integer $\nu(x) \in \mathbb{Z}$ such that $x = \frac{a}{b} p^{\nu(x)}$, with $a, b$ not divisible by $p$. Then, $|x|_p = p^{-\nu(x)}$ is a norm over $\mathbb{Q}$. The $p$-adic field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\,.\,|_p$. Now, $(\mathbb{Q}_p^n, \|\,.\,\|)$, with $\|x\| = \max\{|x_i|_p : 1 \leq i \leq n\}$ is a non-Archimedean normed space over $\mathbb{Q}_p$. More generally, if $\mathbb{K}$ is a non-Archimedean field, then $(\mathbb{K}, \|\,.\,\|)$ is a non-Archimedean normed space over $\mathbb{K}$.*

**Example 2.4.** *Let $X$ be a set and let $\mathbb{K}$ be a non-Archimedean field. The set $\mathcal{B}(X, \mathbb{K})$ of all bounded functions on $X$ with values in $\mathbb{K}$ is a vector space over $\mathbb{K}$, with pointwise addition and scalar multiplication. Equipped with the norm*

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\},$$

*the space $\mathcal{B}(X, \mathbb{K})$ is a non-Archimedean normed space over $\mathbb{K}$.*

A non-Archimedean Banach space is a complete non-Archimedean space, i.e. where each Cauchy sequence is convergent. $\mathbb{K}^n$ and $\mathcal{B}(X, \mathbb{K})$ are examples of non-Archimedean Banach spaces. An important feature of non-Archimedean spaces is the following. Let $\{x_n\}$ be sequence in a non-Archimedean space $(X, \|\cdot\|)$ and let $n > m$. Since

$$\|x_n - x_m\| \leq \max\{\|x_{i+1} - x_i\| : m \leq i \leq n-1\}$$

we conclude that, in striking difference with real analysis, $\{x_n\}$ is a Cauchy sequence if, and only if, $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$.

Throughout this article, $G$ is an abelian group and $(X, \|\cdot\|)$ is a non-Archimedean Banach space. We will adopt the following notation. Given a map $f : G \to X$, we define the Cauchy difference $Cf : G \times G \to X$ to be

$$Cf(x, y) = f(x + y) - f(x) - f(y).$$

The second Cauchy difference $C^2 f : G \times G \times G \to X$ is given by

$$C^2 f(x, y, z) = Cf(x + y, z) - Cf(x, z) - Cf(y, z).$$

We easily see that equation (2) is equivalent to the vanishing of the second Cauchy difference

$$C^2 f(x, y, z) = 0.$$

Note that if we take $x = y = z = 0$ in (2) we conclude that $f(0) = 0$. On the other hand, if $f$ is an even map, taking $z = -y$ we see that $f$ is quadratic.

## 3. Stability of the functional equation

In this section we study the stability of functional equation (2) for two cases: firstly, when the solution is an odd map and secondly when the solution is an even map, with the additional condition that $f(0) = 0$. We begin by recalling the definition of stability in the sense of Hyers-Ulam-Rassias for our setting. Let $G$ be a group and $\psi, \Phi : G^3 \to [0, +\infty)$ be functions satisfying certain conditions. Let $X$ be a non-Archimedean Banach space. If for every function $f : G \to X$ satisfying the inequality

$$\|C^2 f(x, y, z)\| \leq \psi(x, y, z)\ ,\ \ \forall x, y, z \in G$$

there exists a function $\varphi : G \to X$ such that

$$C^2 \varphi(x, y, z) = 0\ ,\ \ \forall x, y, z \in G,$$

and

$$\|f(x) - \varphi(x)\| \leq \Phi(x, x, x)\ ,\ \ \forall x \in G$$

then the functional equation (2) is stable in the sense of Hyers-Ulam-Rassias on $(G, X)$.

### 3.1. The odd case

The first result establishes Hyers-Ulam-Rassias stability for an approximate odd solution of equation (2).

**Theorem 3.1.** *Let $\psi : G \times G \times G \to [0, +\infty)$ be a function such that*

$$\lim_{n \to +\infty} \frac{\psi(2^n x, 2^n y, 2^n z)}{|2|^n} = 0 \tag{4}$$

*for all $x, y, z \in G$. Let for each $x \in G$ the limit*

$$\lim_{n\to+\infty} \max\left\{\frac{\Psi(2^j x)}{|2|^j} : 0 \le j < n\right\} := \widetilde{\psi}_A(x) \tag{5}$$

*exist, where*

$$\Psi(x) = \max\{\psi((-1)^{\varepsilon_1}x, (-1)^{\varepsilon_2}x, (-1)^{\varepsilon_3}x) : \varepsilon_i = 0, 1, \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \equiv 1, 2 \,(\mathrm{mod}\, 3)\}$$

*for all $x \in G$.*

*Suppose $f : G \to X$ is an odd map satisfying*

$$\|C^2 f(x, y, z)\| \le \psi(x, y, z)\,, \tag{6}$$

*for all $x, y, z \in G$. Then, there exist an additive map $A : G \to X$ such that*

$$\|f(x) - A(x)\| \le \frac{1}{|2|}\widetilde{\psi}_A(x), \tag{7}$$

*for all $x \in G$. Moreover, if*

$$\lim_{k\to+\infty}\lim_{n\to+\infty} \max\left\{\frac{\Psi(2^j x)}{|2|^j} : k \le j < n + k\right\} = 0 \tag{8}$$

*then the additive map $A$ is unique.*

*Proof.* Put $x = y = -z$ in (6). We have

$$\|3f(x) - f(2x) + f(-x) - 2f(0)\| \le \psi(x, x, -x)$$

Since $f$ is odd,

$$\|f(2x) - 2f(x)\| \le \psi(x, x, -x), \tag{9}$$

Replacing $x$ by $-x$ in (9) gives

$$\|f(2x) - 2f(x)\| \le \psi(-x, -x, x). \tag{10}$$

It follows that

$$\|f(2x) - 2f(x)\| \le \Psi(x). \tag{11}$$

Replacing $x$ by $2^{n-1}x$ in (11), we obtain

$$\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}}\right\| \le \frac{\Psi(2^{n-1}x)}{|2|^n}. \tag{12}$$

By (4) and (12), the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence. Since $X$ is complete it follows that $\left\{\frac{f(2^n x)}{2^n}\right\}$ is convergent. Define a map $A : G \to X$ by

$$A(x) = \lim_{n\to+\infty}\frac{f(2^n x)}{2^n}\ , \text{ for all } x \in G. \tag{13}$$

By induction we have

$$\left\|\frac{f(2^n x)}{2^n} - f(x)\right\| \leq \frac{1}{|2|}\max\left\{\frac{\Psi(2^j x)}{|2|^j} : 0 \leq j < n\right\}, \tag{14}$$

for all $n \in \mathbb{N}$ and $x \in G$.

In fact, the case $n = 1$ follows from (11). By hypothesis and from (12), we have

$$\begin{aligned}
\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - f(x)\right\| &= \left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n} + \frac{f(2^n x)}{2^n} - f(x)\right\| \\
&\leq \max\left\{\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\right\|, \left\|\frac{f(2^n x)}{2^n} - f(x)\right\|\right\} \\
&\leq \max\left\{\frac{\Psi(2^n x)}{|2|^{n+1}}, \frac{\Psi(2^j x)}{|2|^{j+1}} : 0 \leq j < n\right\} \\
&= \frac{1}{|2|}\max\left\{\frac{\Psi(2^j x)}{|2|^j} : 0 \leq j < n+1\right\}.
\end{aligned}$$

Now, taking the limit as $n \to +\infty$ in (14) and using (5) we obtain (7). Replacing $(x, y, z)$ by $(2^n x, 2^n y, 2^n z)$ in (6) we obtain

$$\left\|\frac{C^2 f(2^n x, 2^n y, 2^n z)}{2^n}\right\| \leq \frac{\psi(2^n x, 2^n y, 2^n z)}{|2|^n}.$$

Therefore, taking the limit as $n \to +\infty$ it follows from (4) that $A$ satisfies equation (2) and is odd since $f$ is odd.

We will now show that $A$ is additive. Set $z = -y$ in (2). Since $A$ is odd, we have

$$A(x+y) + A(x-y) = 2A(x). \tag{15}$$

Interchanging $x$ and $y$ in (15):

$$A(y+x) + A(y-x) = 2A(y). \tag{16}$$

Now, add (15) and (16)

$$2A(x+y) + A(x-y) + A(-(x-y)) = 2A(x) + 2A(y),$$

and since $A$ is odd we conclude that

$$A(x+y) = A(x) + A(y).$$

Finally, suppose $\widetilde{A} : G \to X$ is another additive map which satisfies (7). We have

$$\begin{aligned}
\|A(x) - \widetilde{A}(x)\| &= \lim_{k\to+\infty}\left\|\frac{A(2^k x)}{2^k} - \frac{\widetilde{A}(2^k x)}{2^k}\right\| \\
&= \lim_{k\to+\infty}|2|^{-k}\|A(2^k x) - \widetilde{A}(2^k x)\| \\
&\leq \lim_{k\to+\infty}|2|^{-k}\|A(2^k x) - f(2^k x) + f(2^k x) - \widetilde{A}(2^k x)\| \\
&\leq \lim_{k\to+\infty}|2|^{-k}\max\{\|A(2^k x) - f(2^k x)\|, \|f(2^k x) - \widetilde{A}(2^k x)\|\} \\
&\leq \frac{1}{|2|}\lim_{k\to+\infty}\lim_{n\to+\infty}\max\left\{\frac{\Psi(2^j x)}{|2|^j} : k \leq j < n+k\right\} \\
&= 0.
\end{aligned}$$

□

Recall that a pseudo length function on an abelian group $G$ is a function $\ell : G \to \mathbb{R}$ that verifies, for every $x, y \in G$, the following properties:

(*i*) $\ell(0) = 0$;

(*ii*) $\ell(x + y) \leq \ell(x) + \ell(y)$;

(*iii*) $\ell(-x) = \ell(x)$.

A length function is a pseudo length function such that $\ell(x) = 0$ implies $x = 0$. A pseudo length function is called homogeneous if $\ell(2x) = 2\ell(x)$. If $\ell$ is homogeneous, then $\ell(2^n x) = 2^n \ell(x)$, for every nonnegative integer $n$ and every $x \in G$.

The next result gives a more concrete function $\psi : G \times G \times G \to [0, +\infty)$.

**Corollary 3.2.** *Let $\rho : [0, +\infty) \to [0, +\infty)$ be a function such that*

(*i*) $\rho(2t) \leq \rho(2)\rho(t) \quad (t \geq 0)$;

(*ii*) $\rho(2) < |2|$.

*Let $G$ be an abelian group with a homogeneous pseudo length function $\ell : G \to \mathbb{R}$ and let $\delta > 0$. Suppose $f : G \to X$ is an odd map verifying*

$$\|C^2 f(x, y, z)\| \leq \delta\rho(\ell(x))\rho(\ell(y))\rho(\ell(z)),$$

*for all $x, y, z \in G$. Then, there exist a unique additive map $A : G \to X$ such that*

$$\|f(x) - A(x)\| \leq \frac{\delta}{|2|}\,\rho(\ell(x))^3,$$

*for all $x \in G$.*

*Proof.* Define $\psi : G \times G \times G \to [0, +\infty)$ by

$$\psi(x, y, z) = \delta\rho(\ell(x))\rho(\ell(y))\rho(\ell(z)).$$

We have

$$\begin{aligned}
\psi(2^n x, 2^n y, 2^n z) &= \delta\rho(\ell(2^n x))\rho(\ell(2^n y))\rho(\ell(2^n z)) \\
&= \delta\rho(2^n \ell(x))\rho(2^n \ell(y))\rho(2^n \ell(z)) \\
&\leq \rho(2^n)^3 \delta\rho(\ell(x))\rho(\ell(y))\rho(\ell(z)) \\
&= \rho(2^n)^3 \psi(x, y, z).
\end{aligned}$$

Dividing by $|2|^n$ and taking the limit as $n \to +\infty$,

$$\begin{aligned}
\lim_{n\to+\infty} \frac{\psi(2^n x, 2^n y, 2^n z)}{|2|^n} &\leq \lim_{n\to+\infty} \frac{\rho(2)^{3n}}{|2|^n}\psi(x, y, z) \\
&= \lim_{n\to+\infty} \left(\frac{\rho(2)}{|2|}\right)^{3n} |2|^{2n}\psi(x, y, z) \\
&= 0.
\end{aligned}$$

By definition,

$$\widetilde{\psi}_A(x) = \lim_{n\to+\infty} \max\left\{\frac{\Psi(2^j x)}{|2|^j} : 0 \leq j < n\right\}.$$

And since $\ell(-x) = \ell(x)$ we have

$$\frac{\Psi(2^j x)}{|2|^j} = \delta\,\frac{\rho(\ell(2^j x))^3}{|2|^j}.$$

The inequality

$$\begin{aligned}\frac{\rho(\ell(2^{k+1}x))}{|2|^{k+1}} &= \frac{\rho(2\ell(2^k x))}{|2||2|^k}\\ &\leq \frac{\rho(2)}{|2|}\frac{\rho(\ell(2^k x))}{|2|^k}\\ &\leq \frac{\rho(\ell(2^k x))}{|2|^k}\end{aligned}$$

allows one to conclude that

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\max\left\{\frac{\Psi(2^j x)}{|2|^j} : k \leq j < n+k\right\} = \lim_{k\to+\infty}\frac{\Psi(2^k x)}{|2|^k} = 0.$$

Applying Theorem 3.1, the result follows. $\square$

*3.2. The even case*

We proceed to establish Hyers-Ulam-Rassias stability for an approximate even solution of equation (2). We need to require that such approximate solution vanishes at zero, i.e. $f(0) = 0$.

**Theorem 3.3.** *Let $\psi : G \times G \times G \to [0, +\infty)$ be a function such that*

$$\lim_{n\to+\infty}\frac{\psi(2^n x, 2^n y, 2^n z)}{|4|^n} = 0 \tag{17}$$

*for all $x, y, z \in G$. Let for each $x \in G$ the limit*

$$\lim_{n\to+\infty}\max\left\{\frac{\Psi(2^j x)}{|4|^j} : 0 \leq j < n\right\} := \widetilde{\psi}_Q(x) \tag{18}$$

*exist, where $\Psi$ is defined as in Theorem 3.1:*

$$\Psi(x) = \max\{\psi((-1)^{\varepsilon_1}x, (-1)^{\varepsilon_2}x, (-1)^{\varepsilon_3}x) : \varepsilon_i = 0, 1, \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \equiv 1, 2 \,(\mathrm{mod}\,3)\}$$

*for all $x \in G$.*

*Suppose $f : G \to X$ is an even map with $f(0) = 0$ satisfying*

$$\|C^2 f(x, y, z)\| \leq \psi(x, y, z), \tag{19}$$

*for all $x, y, z \in G$. Then, there exist a quadratic map $Q : G \to X$ such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|}\widetilde{\psi}_Q(x), \tag{20}$$

*for all $x \in G$. Moreover, if*

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\max\left\{\frac{\Psi(2^j x)}{|4|^j} : k \leq j < n+k\right\} = 0 \tag{21}$$

*then the quadratic map $Q$ is unique.*

*Proof.* Put $x = y = -z$ in (19). Since $f$ is even and $f(0) = 0$, we have

$$\|f(2x) - 4f(x)\| \leq \psi(x, x, -x), \tag{22}$$

Replacing $x$ by $-x$ in (22) gives

$$\|f(2x) - 4f(x)\| \leq \psi(-x, -x, x). \tag{23}$$

Hence,

$$\|f(2x) - 4f(x)\| \leq \Psi(x). \tag{24}$$

Replacing $x$ by $2^{n-1}x$ in (24), we get

$$\left\|\frac{f(2^n x)}{4^n} - \frac{f(2^{n-1}x)}{4^{n-1}}\right\| \leq \frac{\Psi(2^{n-1}x)}{|4|^n}. \tag{25}$$

It follows from (17) and (25) that $\left\{\dfrac{f(2^n x)}{4^n}\right\}$ is a Cauchy sequence and, since $X$ is complete, $\left\{\dfrac{f(2^n x)}{4^n}\right\}$ is convergent. Define, for each $x \in G$, a map $Q : G \to X$ by

$$Q(x) = \lim_{n\to+\infty} \frac{f(2^n x)}{4^n}. \tag{26}$$

By induction, similar to Theorem 3.1, we conclude that

$$\left\|\frac{f(2^n x)}{4^n} - f(x)\right\| \leq \frac{1}{|4|} \max\left\{\frac{\Psi(2^j x)}{|4|^j} : 0 \leq j < n\right\}, \tag{27}$$

for all $n \in \mathbb{N}$ and $x \in G$.

Taking the limit as $n \to +\infty$ in (27) and using (18) we obtain (20). Replacing $(x, y, z)$ by $(2^n x, 2^n y, 2^n z)$ in (19) we obtain

$$\left\|\frac{C^2 f(2^n x, 2^n y, 2^n z)}{4^n}\right\| \leq \frac{\psi(2^n x, 2^n y, 2^n z)}{|4|^n}.$$

Taking the limit as $n \to +\infty$ it follows from (17) that $Q$ satisfies equation (2).

To show that $Q$ is quadratic simply substitute $z = -y$ in equation (2) and take into consideration that $Q(0) = 0$ and $Q$ is even since $f$ is even. We have

$$Q(x) + Q(x) + Q(y) + Q(-y) = Q(x + y) + Q(0) + Q(x - y),$$

and so

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y).$$

Finally, admitting condition (21), the proof that $Q$ is the unique quadratic map verifying (20) is similar to Theorem 3.1. □

In analogy with corollary 3.2 we have the following result.

**Corollary 3.4.** *Let* $\rho : [0, +\infty) \to [0, +\infty)$ *be a function such that*

(*i*) $\rho(2t) \leq \rho(2)\rho(t) \quad (t \geq 0)$;

(*ii*) $\rho(2) < |2|^2$.

*Let $G$ be an abelian group with a homogeneous pseudo length function $\ell : G \to \mathbb{R}$ and let $\delta > 0$. Suppose that $f : G \to X$ is an even map verifying $f(0) = 0$ and*

$$\|C^2 f(x, y, z)\| \leq \delta\rho(\ell(x))\rho(\ell(y))\rho(\ell(z)),$$

*for all $x, y, z \in G$. Then, there exist a unique quadratic map $Q : G \to X$ such that*

$$\|f(x) - Q(x)\| \leq \frac{\delta}{|4|}\rho(\ell(x))^3,$$

*for all $x \in G$.*

The proof is analogous to that of Corollary 3.2.

## 4. Main result

Using Theorems 3.1 and 3.3 we may now prove Hyers-Ulam-Rassias stability for equation (2). This is accomplished by using the canonical decomposition of $f$ into the sum of the odd and even parts.

**Theorem 4.1.** *Let $\psi : G \times G \times G \to [0, +\infty)$ be a function such that*

$$\lim_{n\to+\infty} \frac{\psi(2^n x, 2^n y, 2^n z)}{|4|^n} = 0$$

*for all $x, y, z \in G$. Let, for each $x \in G$, $\widetilde{\psi}_A(x)$ and $\widetilde{\psi}_Q(x)$ exist as defined in Theorems 3.1 and 3.3.*
*Let $f : G \to X$ be a map such that $f(0) = 0$ satisfying*

$$\|C^2 f(x, y, z)\| \leq \psi(x, y, z), \tag{28}$$

*for all $x, y, z \in G$. Then, there exist an additive map $A : G \to X$ and a quadratic map $Q : G \to X$ such that*

$$\|f(x) - A(x) - Q(x)\| \leq \frac{1}{|4|}\max\left\{\widetilde{\psi}_A(x), \widetilde{\psi}_A(-x), \frac{\widetilde{\psi}_Q(x)}{|2|}, \frac{\widetilde{\psi}_Q(-x)}{|2|}\right\}, \tag{29}$$

*for all $x \in G$. Moreover, if*

$$\lim_{k\to+\infty}\lim_{n\to+\infty} \max\left\{\frac{\Psi(2^j x)}{|4|^j} : k \leq j < n + k\right\} = 0 \tag{30}$$

*then the additive map $A$ and the quadratic map $Q$ are unique.*

*Proof.* Notice that, since the ground field is non-Archimedean we have

$$|4| = |2||2| \leq |2|.$$

So, $|4|^n \leq |2|^n$ and condition (17) implies condition (4).
Write $f(x) = f_o(x) + f_e(x)$, where

$$f_o(x) = \frac{f(x) - f(-x)}{2} \quad \text{and} \quad f_e(x) = \frac{f(x) + f(-x)}{2}$$

are the odd and even part of $f$, respectively. We have

$$\begin{aligned}\|C^2 f_o(x, y, z)\| &= \frac{1}{|2|}\|C^2 f(x, y, z) - C^2 f(-x, -y, -z)\| \\ &\leq \frac{1}{|2|}\max\{\|\psi(x, y, z)\|, \|\psi(-x, -y, -z)\|\}\end{aligned}$$

and

$$\begin{aligned}\|C^2 f_e(x,y,z)\| &= \frac{1}{|2|}\|C^2 f(x,y,z) + C^2 f(-x,-y,-z)\| \\ &\leq \frac{1}{|2|}\max\{\|\psi(x,y,z)\|, \|\psi(-x,-y,-z)\|\}\end{aligned}$$

By Theorem 3.1 there exist an additive map $A : G \to X$ such that

$$\|f_o(x) - A(x)\| \leq \frac{1}{|2|^2}\max\{\widetilde{\psi_A}(x), \widetilde{\psi_A}(-x)\}$$

for every $x \in G$. By Theorem 3.3 there exist a quadratic map $Q : G \to X$ such that

$$\|f_e(x) - Q(x)\| \leq \frac{1}{|2|^3}\max\{\widetilde{\psi_Q}(x), \widetilde{\psi_Q}(-x)\}.$$

for every $x \in G$. It follows that

$$\begin{aligned}\|f(x) - A(x) - Q(x)\| &= \|f_o(x) - A(x) + f_e(x) - Q(x)\| \\ &\leq \max\{\|f_o(x) - A(x)\|, \|f_e(x) - Q(x)\|\} \\ &\leq \frac{1}{|4|}\max\left\{\widetilde{\psi_A}(x), \widetilde{\psi_A}(-x), \frac{\widetilde{\psi_Q}(x)}{|2|}, \frac{\widetilde{\psi_Q}(-x)}{|2|}\right\}.\end{aligned}$$

The proofs of the uniqueness of $A$ and $Q$, admitting condition (30), are similar to the one in Theorem 3.1. □

**Corollary 4.2.** *Let $\rho : [0,+\infty) \to [0,+\infty)$ be a function such that*

(*i*) $\rho(2t) \leq \rho(2)\rho(t) \quad (t \geq 0)$;

(*ii*) $\rho(2) < |2|^2$.

*Let $G$ be an abelian group with a homogeneous pseudo length function $\ell : G \to \mathbb{R}$ and let $\delta > 0$. Suppose that $f : G \to X$ is a map verifying $f(0) = 0$ and*

$$\|C^2 f(x,y,z)\| \leq \delta\rho(\ell(x))\rho(\ell(y))\rho(\ell(z)),$$

*for all $x, y, z \in G$. Then, there exist a unique additive map $A : G \to X$ and a unique quadratic map $Q : G \to X$ such that*

$$\|f(x) - A(x) - Q(x)\| \leq \frac{\delta}{|4|}\rho(\ell(x))^3,$$

*for all $x \in G$.*

*Proof.* From Theorem 4.1, and Corollaries 3.2 and 3.4, we have

$$\begin{aligned}\|f(x) - A(x) - Q(x)\| &= \|f_o(x) - A(x) + f_e(x) - Q(x)\| \\ &\leq \max\{\|f_o(x) - A(x)\|, \|f_e(x) - Q(x)\|\} \\ &\leq \frac{\delta}{|4|}\rho(\ell(x))^3,\end{aligned}$$

for all $x \in G$. □

**Example 4.3.** *The control function $\psi : G^3 \to [0,+\infty)$ appearing in Theorems 3.1, 3.3, and 4.1 was made explicit in Corollaries 3.2, 3.4 and 4.2, in terms of a homogeneous pseudo length function $\ell : G \to \mathbb{R}$ and a certain real function $\rho : [0,+\infty) \to [0,+\infty)$. We now give a simple example of this function $\rho$. Given $p > 1$, fix $0 < \alpha_p < 1$ such that $\alpha_p < |2|^2 2^{-p}$. Then we can take $\rho(t) = \alpha_p t^p$. Since $|2|^2 < |2| \leq 1$ such map $\rho$ serve as an example for the above mentioned Corollaries.*

According to [18, Theorem 1.2], given a homogeneous pseudo length function $\ell : G \to [0, +\infty[$, there exist a real Banach space $(B, \|\,.\,\|)$ and a group homomorphism $\phi : G \to B$ such that $\ell(x) = \|\phi(x)\|$, for all $x \in G$. Moreover, if $\ell$ is a length function, we can take $\phi$ to be an isometric embedding. This shows we could work directly with a real Banach space in Corollaries 3.2, 3.4 and 4.2 instead of an abelian group endowed with a homogeneous length function.

**Acknowledgements**

The authors would like to thank the anonymous referees for the careful reading of the manuscript and the valuable comments which helped to improve our work.

## References

[1] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.
[2] P. Erdös, S. Shelah, Separability properties of almost-disjoint families of sets, Israel Journal of Mathematics 12 (1972) 207–214.
[3] L. Arriola, W. Beyer, Stability of the Cauchy functional equation over p-adic fields, Real Analysis Exchange 31:1 (2005/2006) 125–132.
[4] M. Fréchet, Sur la definition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert, Annals of Mathematics (2) 36 (1935) no. 3 705–718.
[5] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, Journal of Mathematical Analysis and Applications 184 (1994) 431–436.
[6] M. Gordji, R. Khodabakhsh, H. Khodaei, et al., A functional equation related to inner product spaces in non-Archimedean normed spaces, Advances in Difference Equations (2011) 2011:37.
[7] D. Hyers, On the stability of the linear functional equation, Proceedings of the National Academy of Sciences of the United States of America 27 (1941) 222–224.
[8] P. Jordan, J. von Neumann, On inner products in linear metric spaces. Annals of Mathematics (2) 36 (1935) no. 3 719–723.
[9] P. Kannappan, Quadratic functional equation and inner product spaces, Results in Mathematics 27 (1995) 368–372.
[10] G. Kim, On the stability of the quadratic mapping in normed spaces, International Journal of Mathematics and Mathematical Sciences, 25:4 (2001) 217–229.
[11] B. V. Senthil Kumar, H. Dutta, Non-Archimedean stability of a generalized reciprocal-quadratic functional equation in several variables by direct and fixed point methods, Filomat 32:9 (2018) 3199–3209.
[12] B. V. Senthil Kumar, H. Dutta, Approximation of multiplicative inverse undecic and duodecic functional equations, Mathematical Methods in the Applied Sciences 42 (2019) 1073–1081.
[13] B. V. Senthil Kumar, H. Dutta, Fuzzy stability of a rational functional equation and its relevance to system design, International Journal of General Systems 48:2 (2019) 157–169.
[14] B. V. Senthil Kumar, H. Dutta, S. Sabarinathan, Approximation of a system of rational functional equations of three variables, International Journal of Applied and Computational Mathematics, 5:39 (2019).
[15] Y.-H. Lee, S.-M. Jung, A fixed point approach to the stability of a general quartic functional equation, 20:3 (2020) 207–215.
[16] M. Moslehian, T. Rassias, Stability of functional equations in non-Archimedean spaces, Applicable Analysis and Discrete Mathematic 1 325–334 (2007).
[17] M. Moslehian, T. Rassias, A characterization of inner product spaces, Kochi Journal of Mathematics 6 (2011) 101–107.
[18] D.H.J. Polymath, Homogeneous length functions on groups, Algebra and Number Theory 12:7 (2018) 1773–1786.
[19] T. Rassias, On the stability of the linear mapping in Banach spaces, Proceedings of the American Mathematical Society 72 (1978) 297–300.
[20] T. Rassias, J. Brzdęk, Editors, Functional Equations in Mathematical Analysis, Springer Optimization and Its Applications 52, Springer, 2012.
[21] N. Sene, Global asymptotic stability of the fractional differential equations, Journal of Nonlinear Sciences and Applications 13 (2020) 171–175.
[22] K. Shah et al., Monotone iterative techniques together with Hyers-Ulam-Rassias stability, Mathematical Methods in the Applied Sciences, Special Issue (2019) 1–18.
[23] S.-M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, Journal of Mathematical Analysis and Applications 222 (1998) 126–137.
[24] S.-M. Jung, Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis, Springer, New York, 2011.
[25] Henrik Stetkaer, Functional Equations on Groups, World Scientific Publishing, Singapore, 2013.
[26] S. Ulam, Problems in Modern Mathematics, Wiley, New York, 1960.